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Microbuckle initiation from a patch of large amplitude fibre waviness in a composite under compression and bending

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Abstract – A finite element couple stress formulation is used to predict microbuckle initiation from a patch of fibre waviness in a unidirectional fibre composite under remote compression and bending. Attention is focused on the knock-down in strength due to large amplitude waviness, with the effects of the physical size of the imperfection included by incorporating the fibre bending resistance within the formulation. The predicted strengths deviate significantly from the simpler kinking theory which neglects the role of fibre bending. Initial imperfections in the form of an infinite band and a circular wavy patch are considered: when these imperfections are of large spatial extent and possess a large misalignment angle, the compressive strength approximates the steady state band broadening stress for an infinite band. The effect of an imposed spatial gradient of stress within the composite is explored by determining the compressive strength of beams of finite height B for the loading cases of pure bending and axial compression. It is found that the compressive strength is sensitive to the magnitude of the imposed stress gradient: the compressive strength of the outer fibres of the beam in bending increases with diminishing height of the beam. This size dependence is much reduced for the case of uniform compression. © 2001 Éditions scientifiques et médicales Elsevier SAS

compressive strength / composites / finite elements / microbuckling

1. Introduction

Long fibre composites typically fail in compression at lower stress levels than in tension, due to the phenomenon of *imperfection-sensitive plastic microbuckling*: compressive collapse occurs by shearing of the matrix between fibres at locations of initial fibre misalignment. Early studies of compressive failure (e.g. (Rosen, 1965)) suggested that failure occurs by an elastic shear bifurcation at an axial applied stress equal to the in-plane shear modulus, G. However, experiments reveal strengths on the order of $G/4$ due to matrix non-linearity and initial fibre misalignment, see for example, Budiansky and Fleck (1993), Fleck(1997), Moran et al. (1995), Kyriakides et al. (1995), Kyriakides and Ruff (1997) and Schapery (1995).

Analytical formulae for imperfection-sensitive plastic microbuckling have been obtained only for the one-dimensional case of an infinite band of initial fibre waviness (Budiansky, 1983; Budiansky and Fleck, 1993; Fleck et al., 1995; Budiansky et al., 1998; Wisnom, 1990). These infinite band solutions can be classified into two categories: *kinking theory*, where the fibre bending resistance is neglected, and *bending theory*, where the fibre bending resistance is included. The bending theory of Fleck et al. (1995) includes a material length scale within the formulation, specified by the fibre diameter d, and is capable of predicting the effect of the physical size of imperfection upon the compressive strength; this theory sits within the framework of couple stress theory. Fleck et al. (1995) thereby found that the compressive strength is moderately sensitive to the width of the infinite band of misaligned fibres: when the width exceeds about $20d$, the compressive strength according to bending theory is comparable to that given by the simpler kinking theory.

More recently, finite element methods have been used to predict the compressive strength of a composite, assuming a two dimensional distribution of initial fibre misalignment. For example, Kyriakides et al. (1995)

performed a finite element analysis of the initiation and growth of microbuckling from a small region of fibre misalignment; they modelled the composite as alternating, perfectly bonded layers of fibres and matrix. In similar fashion, Sutcliffe et al. (1996) modelled microbuckle initiation and early growth from a sharp open notch under remote compressive loading. This approach is useful when the initial region of fibre waviness extends over a small number of fibres, but is impractical in terms of computer time when a large number of fibres is considered. Alternatively, Fleck and Shu (1995) generalised the one-dimensional analysis of Fleck et al. (1995), and treated the composite as a 2D Cosserat continuum with a bending resistance set by the fibre diameter d. Fleck and Shu also developed a finite element code to address fibre microbuckling, and thereby predicted the compressive strength associated with an elliptical region of initial fibre misalignment under multi-axial loading (Fleck and Shu, 1995; Shu and Fleck, 1997). It was found that the dominant geometrical feature is the magnitude of initial fibre rotation, and the length ℓ of the initial imperfection in the transverse direction. The compressive strength decreases with increasing ℓ/d from the elastic bifurcation value G at $\ell/d = 0$ (Rosen, 1965) to the infinite band compressive strength at large ℓ/d. The infinite band analysis is adequate provided ℓ exceeds about $400d$.

Scope of paper

To date, the compressive strength of unidirectional composites has been calculated only for wavy patches of small misalignment angle (on the order of a few degrees). However, the initial fibre waviness in woven

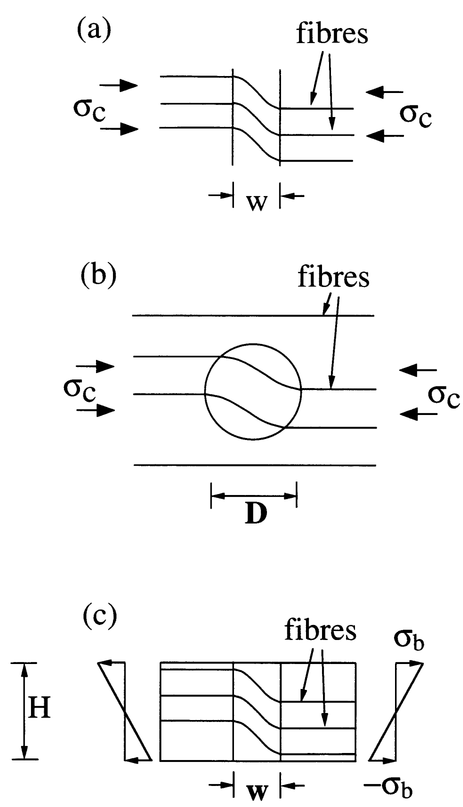


Figure 1. Geometries of imperfection considered in a unidirectional fibre composite: (a) an infinite band of waviness; (b) a circular patch of waviness, and (c) a beam containing a parallel-sided band of waviness under bending.

composites and in through-thickness stitched laminates is often as large as 10°. In this paper, both kinking and bending theories of microbuckling are used to predict the compressive strength due to an infinite band of large fibre misalignment (*figure 1(a)*). Bending theory is also used to determine the compressive strength for a finite circular patch of large fibre waviness (*figure 1(b)*).

Compressive failure of a unidirectional fibre composite beam under bending

Wisnom et al. (1997) have conducted bending tests on unidirectional T800/924 beams in order to measure the effect of beam height H on the compressive failure strain. They observed that the compressive failure strain on the outermost fibre of the beam increased by 50% when the height H was decreased from 8 mm to 1 mm. These observations support the notion that the compressive strength of a composite depends upon the imposed stress gradient. Wisnom (1994) has also made predictions of the effect of beam height upon the microbuckling strength by performing a non-linear finite element analysis of a unidirectional composite beam under pure bending. Two-noded beam elements were used to represent the axial and bending stiffness of the fibres, and four-noded plane stress continuum elements were used to represent the transverse and shear properties of the matrix. The beam elements were given a sinusoidal misalignment along the sides of the continuum elements in order to represent pre-existing fibre waviness (the maximum fibre misalignment angle was taken as 2° and the wavelength was in the range 0.5–1 mm). Wisnom found that the compressive bending strength increased by about 50% when the beam height H was reduced from 8 mm to 1 mm, in support of his experimental observations. In the current paper, the effect of a spatial gradient of stress is explored further: predictions are made for the effect of beam height H upon the bending strength and the uniaxial compressive strength (*figure 1(c)*).

2. Kinking theory

Fleck and Budiansky (1991) have derived the kinematic and equilibrium relations for kinking within a band of finite width and infinite extent, and oriented at an angle β to the overall fibre direction, as shown in *figure 2(a)*. The strain state is derived assuming that the fibres are inextensional and rotate through an additional angle ϕ under load from an initial misalignment angle $\bar{\phi}$ in the stress-free configuration. Budiansky and Fleck (1993) made use of these relations for the case of small misalignments $\bar{\phi}$ and small additional rotations ϕ, and derived an analytic expression for the compressive strength of a composite upon making a number of constitutive assumptions. In the current work, we shall explore the accuracy of the analytical results of Budiansky and Fleck (1993) by performing full numerical analysis of the governing relations of Budiansky and Fleck (1993) for the case where small angles are not assumed. We begin by restating the governing kinematic and equilibrium relations for kinking, and the assumed constitutive response of the composite within the kink band.

2.1. Kinking: equilibrium and kinematics

The kinking theory proposed by Budiansky and Fleck (1993) is in the spirit of an infinite-band one-dimensional shear-localisation analysis. A uniform imperfection in the form of a finite fibre misalignment angle is assumed within a band (*figure 2(a)*) and the evolution of fibre rotation within the band is deduced from algebraic relations for the continuity of traction and displacement across the band boundary.

Consider the collapse of a kink band inclined at an angle β to the main fibre direction, as shown in *figure 2(a)*. It is assumed that the fibres are inextensional and that the uniform strain within the kink band is associated with the fibre rotation ϕ additional to the initial fibre misalignment $\bar{\phi}$. On application of an axial stress σ_L^∞

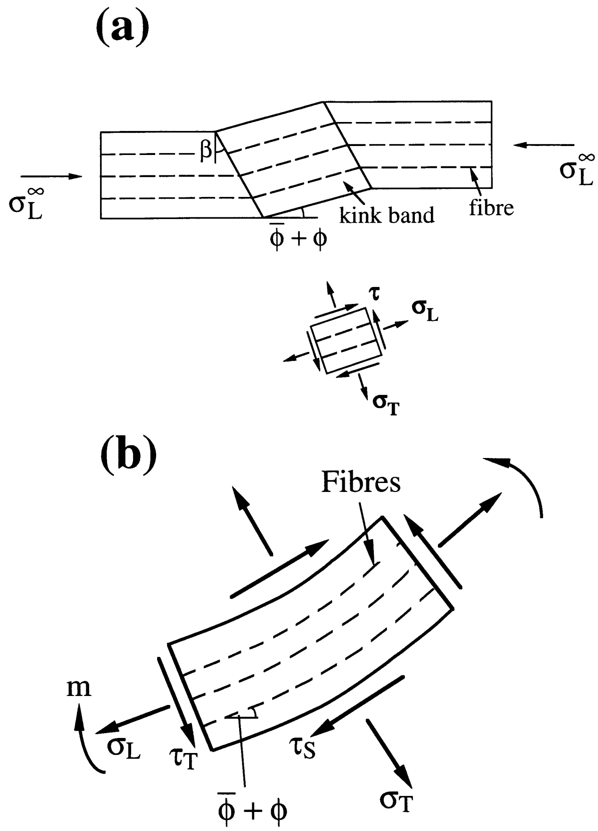


Figure 2. (a) Stress state within a kink band; (b) Stress state of composite by bending theory. The presence of a couple stress *m* gives rise to a difference between the sliding shear stress τ_S and the transverse shear stress τ_T.

to the composite, the stress state within the band comprises an axial stress σ_L along the local fibre direction, a transverse stress σ_T and a longitudinal shear stress shear stress τ, as defined in *figure 2(a)*. Continuity of traction across the band boundary provides

$$-\sigma_L^\infty \cos\beta \cos(\bar{\phi}+\phi) = \sigma_L \cos(\beta-\bar{\phi}-\phi)\cos(\bar{\phi}+\phi) + \tau \sin(\beta-\bar{\phi}-\phi) \tag{2.1}$$

and

$$-\sigma_L^\infty \cos\beta \sin(\bar{\phi}+\phi) = \sigma_T \sin(\beta-\bar{\phi}-\phi) + \tau \cos(\beta-\bar{\phi}-\phi). \tag{2.2}$$

During collapse, it is assumed that fibres within the band rotate at a rate $\dot{\phi}$, thereby inducing a shear strain rate $\dot{\gamma}$, given by

$$\dot{\gamma} = \dot{\phi} \tag{2.3a}$$

and a transverse strain rate $\dot{e}_T$, given by

$$\dot{e}_T \;=\; \dot{\phi}\tan(\beta-\bar{\phi}-\phi). \tag{2.3b}$$

These relations can be integrated immediately, to give,

$$\gamma = \phi \tag{2.4a}$$

and

$$e_T = \log\left[\frac{\cos(\beta - \bar{\phi} - \phi)}{\cos(\beta - \bar{\phi})}\right]. \tag{2.4b}$$

It is assumed that deformation is limited to the kink band and that the kink band is of infinite length (and finite width); consequently the band inclination β remains fixed during collapse. It is reasonable to make the assumption that material outside the kink band remains rigid for the case of a unidirectional composite comprising stiff fibres and subjected to remote axial loading; the cases of in-plane remote shear loading or transverse tension are not considered in the present study.

2.2. Deformation theory constitutive law

It remains to stipulate a constitutive law to relate the stress and strain measures within the kink band. Here, we shall employ a deformation theory constitutive law as proposed by Budiansky and Fleck (1993), and as justified by the experiments of Fleck and Jelf (1995). The transverse stress σ_T and shear stress τ depend upon (e_T, γ) according to

$$\sigma_T = R^2 G_S(\tau_e) e_T \tag{2.5a}$$

and

$$\tau = G_s(\tau_e)\gamma, \tag{2.5b}$$

where the material constant R is defined by $R^2 \equiv E_T/G$, E_T is the transverse Young's modulus and G is the in-plane longitudinal shear modulus of the composite. The effective stress τ_e and the effective strain γ_e are defined by

$$\tau_e \equiv \sqrt{\tau^2 + \left(\frac{\sigma_T}{R}\right)^2} \tag{2.6a}$$

and

$$\gamma_e \equiv \sqrt{\gamma^2 + R^2 e_T^2} \tag{2.6b}$$

respectively, and are related by

$$\gamma_e = \frac{\tau_e}{G_s(\tau_e)} \tag{2.7}$$

such that $G_s(\tau_e)$ is the secant shear modulus in a simple shear test at a shear stress level τ_e. It is convenient to use the empirical Ramberg–Osgood law to describe the shape of the shear stress versus strain curve for typical engineering composites,

$$\frac{\gamma_e}{\gamma_Y} = \frac{\tau_e}{\tau_Y} + \frac{3}{7}\left(\frac{\tau_e}{\tau_Y}\right)^n, \tag{2.8}$$

where (τ_Y, γ_Y, n) are taken as material constants. (For polymer matrix composites, τ_Y is in the range 40–60 MPa, γ_Y equals approximately 1% and the strain-hardening exponent n is in the range 3–10, as collated by Fleck and Jelf (1995).) The shear yield stress τ_Y and shear yield strain γ_Y are related via $\tau_Y = G\gamma_Y$. This gives a secant modulus $G_s \equiv \tau_e/\gamma_e$ of

$$G_s(\tau_e) = G\left(1 + \frac{3}{7}\left(\frac{\tau_e}{\tau_Y}\right)^{n-1}\right)^{-1}. \tag{2.9}$$

Role of volumetric lock-up

The volumetric strain within the kink band equals the transverse strain e_T since the fibres are taken to be inextensional and plane strain conditions apply. Experimental evidence (for example, Fleck and Jelf (1995)) suggests that a tensile transverse strain e_T is accommodated by microcracking of the matrix. As the fibres rotate and ϕ increases during collapse, e_T first increases, attains a maximum at $\phi = \beta - \bar{\phi}$ and then decreases through zero at $\phi = 2(\beta - \bar{\phi})$. Thus, we anticipate that the microcracks close and 'volumetric lock-up' occurs when e_T vanishes at $\phi = 2(\beta - \bar{\phi})$. The subsequent transverse response for $e_T < 0$ is taken to be linear elastic,

$$\sigma_T = E_T e_T \tag{2.10}$$

while the shear response continues to be non-linear, such that

$$\tau_e = |\tau| \quad \text{and} \quad \gamma_e = |\gamma|, \tag{2.11}$$

and the $\tau_e - \gamma_e$ relation is specified by (2.8). This state is labelled the 'locked-up state'. In the current study, the significance of lock-up is explored by also considering the 'no lock-up' case wherein (2.5)–(2.6) are maintained for $e_T < 0$.

It is emphasised that the focus of the current study is the initiation of fibre microbuckling by small additional fibre rotations ϕ from large initial waviness, and not the precise details of fibre lock-up at large fibre rotations. It is envisaged that, as the fibres rotate from the initial stress free but wavy configuration, tensile microcracking of the matrix occurs; this is supported by the recent experimental and theoretical study of Harte and Fleck (1999) on the mechanics of fibre rotation within a $\pm\theta$ braid. Harte and Fleck (1999) subjected a glass-epoxy braid to axial tension, in order to measure the evolution of matrix microcracking as the fibres scissor; the underlying mechanics are very similar to that observed in fibre microbuckling. On extension of the braid, the fibres rotated, with associated tensile microcracking of the matrix. With further rotation of the fibres the matrix underwent continued shearing parallel to the local fibre direction and compressive transverse straining causing the microcracks to coalesce and form a rubble layer. No sudden point of volumetric lock-up was observed, but large compressive transverse stresses developed when the net volumetric strain within the composite became negative (at a fibre rotation ϕ of about 2β).

The phenomenon of lock-up is included implicitly in the analyses of Jensen (1999) and Hsu et al. (1999): they assume an elastic-incompressible plastic composite response, so that as the fibres rotate through 2β there is a build-up of compressive stress transverse to the current fibre direction within the band. The studies of Budiansky and Fleck (1993), Fleck (1997) and that given here assume a plastically dilatant matrix (due to microcracking) and so it is necessary to include the lock-up behaviour explicitly.

2.3. Solution strategy

The collapse response is obtained in the form of σ_L^∞ as a function of fibre rotation ϕ, for a uniform initial waviness $\bar{\phi} \equiv \bar{\phi}_0$ within a band of inclination β. In general, a maximum in σ_L^∞ occurs after an additional fibre rotation ϕ of only a few degrees: the compressive strength is obtained by solving the system of non-linear equations (2.1)–(2.11) by the Newton–Raphson method.

3. 2D couple stress theory and the finite element model

In addition to the use of kinking theory, a couple stress formulation is used to predict the compressive strength due to microbuckling from an infinite band of waviness (*figure 1(a)*) and from regions of fibre misalignment (*figure 1(b),(c)*). The linear elastic fibres are assumed to carry couple stress, whilst the matrix is assumed to deform in a non-linear manner under shear and transverse stress, as measured directly by Fleck and Jelf (1995). A representative material element in the deformed configuration is shown in *figure 2(b)*. The element is subjected to a longitudinal compressive stress component σ_L aligned with the fibre direction, and a transverse tensile stress σ_T. The stress tensor is no longer symmetric, due to the presence of a bending moment m per unit current area (couple stress) borne by the fibres, and so the shear stress τ of kinking theory is replaced by a sliding shear stress τ_S along the fibre direction, and a shear stress τ_T in the transverse direction.

The fibres are modelled as Timoshenko beams and deform in bending and in shear. Thus, the cross-section of each fibre is assumed to rotate by an angle θ_f, which in general is different from the rotation of the neutral axis of the fibre ϕ. A Lagrangian formulation is employed to describe the deformed configuration in terms of the initial reference configuration. In the following section, we shall briefly summarise the deformation theory and flow theory constitutive laws of Fleck and Shu (1995) and Shu and Fleck (1997), respectively, and then describe how the finite element method is implemented to investigate the two-dimensional response of a fibre composite. Full details are given in Fleck and Shu (1995) and in Shu and Fleck (1997).

3.1. Constitutive law

The couple stress m within a representative material element is related to the fibre curvature $\kappa = \mathrm{d}\theta_f/\mathrm{d}s$ by

$$m = \frac{E_L d^2}{16}\kappa, \tag{3.1}$$

where s is the arc-length along the fibre in the deformed configuration, and E_L is the longitudinal modulus of the composite. Likewise, the longitudinal stress in the composite is given by the linear elastic relation

$$\sigma_L = E_L e_L, \tag{3.2}$$

where e_L is the longitudinal elastic strain of the composite.

It is assumed that the composite deforms in a plane strain manner, and has a non-linear shear and transverse response, in accordance with a deformation theory version of plasticity (Shu and Fleck, 1997). Some additional calculations are performed using a flow theory version of plasticity, as laid down by Fleck and Shu (1995). The main details of these plasticity laws are summarised below. With axes aligned with the current fibre direction, the composite suffers a sliding shear strain rate $\dot{\gamma}_S$ and a transverse strain rate $\dot{e}_T$ associated with the rates of sliding shear stress $\dot{\tau}_S$ and transverse stress $\dot{\sigma}_T$.

For both the deformation and flow theory versions of the theory, an effective shear stress τ_e is defined in terms of the shear stress τ_S and the transverse stress σ_T by an alternative version of (2.6a),

$$\tau_e \equiv \sqrt{\tau_S^2 + \left(\frac{\sigma_T}{R}\right)^2}, \tag{3.3}$$

where R is again the ratio of transverse yield strength to shear yield strength of the composite. For the deformation theory version, the effective strain γ_e reads

$$\gamma_e \equiv \sqrt{\gamma_S^2 + R^2 \mathrm{e}_T^2} \tag{3.4}$$

whilst for the flow theory version the effective strain rate $\dot{\gamma}_e$ is specified by

$$\dot{\gamma}_e \equiv \sqrt{\dot{\gamma}_S^2 + R^2 \dot{\varepsilon}_T^2}. \tag{3.5}$$

For both the deformation and flow theory versions, the effective stress is related to the effective strain by the generalised Ramberg–Osgood law (2.8). After lock-up, the constitutive relations (2.10)–(2.11) are enforced, with the role of τ replaced by τ_S, and γ by γ_S.

3.2. *Finite element implementation*

The finite element procedure requires an expression for the global tangent stiffness matrix of the structure. This stiffness matrix is obtained from the rate form of virtual work for the governing field relations; the full expression for the rate virtual work is given in (Fleck and Shu, 1995), and is omitted here for the sake of brevity. It is derived from the following virtual work statement,

$$\int_V [\sigma_L \delta\varepsilon_L + \sigma_T \delta\varepsilon_T + \tau_S \delta\gamma_S + \tau_T \delta\gamma_T + m\delta\kappa]\, \mathrm{d}V = \int_S [t_i \delta u_i + q\delta\theta_f]\, \mathrm{d}S, \tag{3.6}$$

where the internal virtual work is calculated over the current volume V, and the external virtual work on the right hand side is integrated over the current boundary S of the solid. The stress traction t_i $(i = 1, 2)$ and the couple stress traction q are in equilibrium with the interior stress field and perform work through the displacement increments $\delta u_i (i = 1, 2)$ and the rotation increment $\delta\theta_f$, respectively. The relation (3.6) is a statement of general Cosserat theory, and the rotation θ_f of the fibre cross-section is treated as an independent kinematic degree of freedom in addition to the two in-plane displacements u_i $(i = 1, 2)$.

Six-noded triangular elements are employed, with 3 degrees of freedom at each node (two displacements and one rotation). The finite element procedure is based upon a Lagrangian formulation of the general finite deformation of the composite, and can deal with both geometrical and material non-linearities. A version of the modified Riks algorithm is adopted to handle the snap-back behaviour associated with the microbuckling response (Crisfield, 1991). An imperfection in the form of a spatial distribution of initial fibre misalignment $\bar{\phi}$ is included within the formulation.

Specification of composite properties

In subsequent sections we focus on the compressive strength of unidirectional carbon fibre reinforced epoxy. Unless otherwise stated, we assume that the fibre volume fraction c is uniform with $c = 0.6$ and the ratio of transverse strength to shear strength is $R = \sqrt{2}$. The longitudinal elastic modulus E_L equals 2000 times the shear yield strength τ_Y; the transverse modulus is $E_T = 200\tau_Y$, the shear modulus of the fibres is $G_f = 200\tau_Y$ and the shear modulus of the composite is $G = 100\tau_Y$. The strain hardening index n is ascribed the value 3 or 10.

4. Results for imperfections of large misalignment amplitude under uniaxial compression

Consider a composite plate with axis x_1 aligned with the fibre direction, and axis x_2 aligned with the transverse direction. Unless otherwise stated, the material properties are taken to be those specified at the end of section 3.2. We first examine the compressive strength associated with microbuckling from an infinite band and from a finite patch of waviness of large misalignment amplitude under remote compression. Second, we examine the effect of fibre misalignment on the compressive strength of a composite beam in bending.

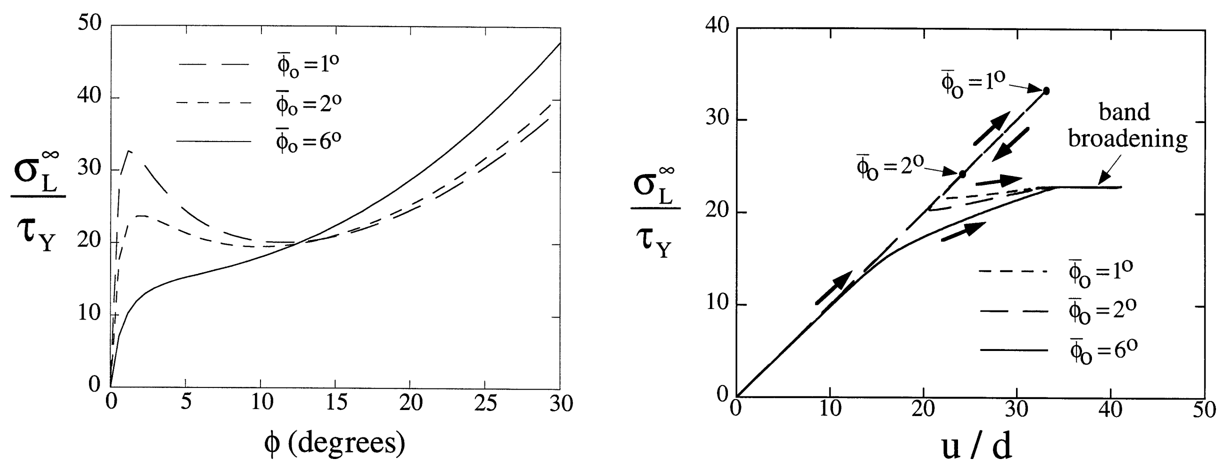
4.1. *Microbuckle initiation from an infinite band of waviness*

We begin by examining the compressive strength of a unidirectional composite containing an infinite band of initial misalignment. The strength is predicted using the deformation theory version of both kinking and bending theories, and the significance of volumetric lock-up is assessed. Consider first kinking theory.

It is recalled that kinking theory contains no material length scale, and so the compressive strength depends only upon the magnitude of the initial fibre misalignment $\bar{\phi}_0$ within the infinite band, in addition to the band orientation β and the constitutive parameters (τ_Y, γ_Y, n) as defined in (2.8). Unless otherwise stated, we shall assume $\beta = 0$, with $\gamma_Y = 1\%$ and $n = 3$. Outside the kink band, the fibre misalignment is assumed to vanish. Typical predictions of the σ_L^∞ versus ϕ collapse response are given in *figure 3(a)*, for the selected values $\bar{\phi}_0 = 1°$, $2°$ and $6°$. The occurrence of lock-up is included in the analysis, and it is noted that lock-up occurs immediately for all values of $\bar{\phi}_0$ since $\beta = 0°$. For small values of $\bar{\phi}_0$ ($= 1°$ and $2°$) the remote axial stress σ_L^∞ attains a local maximum value σ_c after an additional fibre rotation ϕ of a few degrees; in contrast, at larger values of $\bar{\phi}_0$ no such peak in stress is evident and σ_L^∞ increases monotonically with increasing ϕ to the maximum achievable value of the Rosen elastic bifurcation stress $\sigma_L^\infty = G = 100\tau_Y$. This prediction is unrealistic, and we need bending theory to model the collapse response at large fibre rotations.

The peak strength is replotted as a function of $\bar{\phi}_0$ in *figure 4*, for the cases where lock-up is included and is ignored. For the case of kinking theory and small $\bar{\phi}_0$, the compressive strength denotes the first local maximum in the stress versus fibre rotation response. The plateau values of σ_c for kinking give the asymptotic strength $\sigma_c = G$, since no local maximum in collapse response is predicted. For small values of $\bar{\phi}_0$, of less than about 3.5°, lock-up has only a minor effect upon the response, and σ_c is given to good accuracy by the small-rotation analytic result of Budiansky and Fleck (1993),

$$\frac{\sigma_c}{G^*} = \frac{1}{1 + n\left(\frac{3}{7}\right)^{1/n}\left[\frac{\bar{\phi}/\gamma_Y^*}{n-1}\right]^{(n-1)/n}}, \tag{4.1}$$



(a) (b)

Figure 3. Collapse response due to an infinite band of fibre misalignment according to (a) kinking theory and (b) bending theory. The results for kinking theory are plotted in the form of axial stress σ_L^∞ versus additional fibre rotation ϕ within the band, whereas for bending theory σ_L^∞ is plotted against the axial shortening u, for a width of imperfection $w = 400d$. In both figures, the deformation theory version of the constitutive law is used, with a strain hardening exponent $n = 3$; the band is transverse to the overall fibre direction, $\beta = 0$.

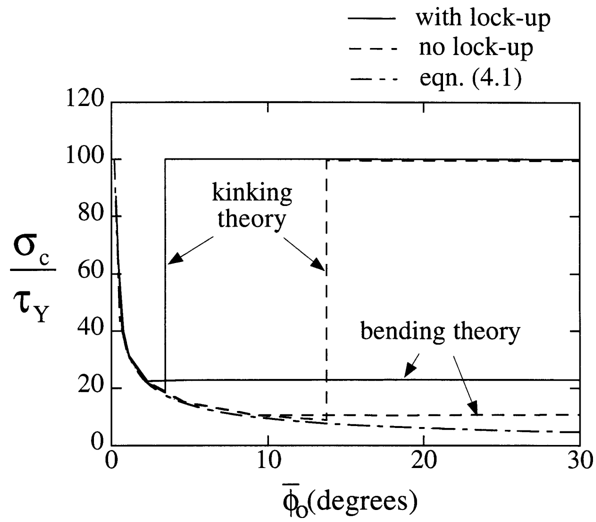


Figure 4. Effect of the amplitude of fibre misalignment $\bar{\phi}_0$ on compressive strength due to an infinite band of fibre waviness by both bending and kinking theories. Deformation theory with $n = 3$, $w = 400d$ and $\beta = 0°$. For the case of kinking theory and small $\bar{\phi}_0$, the compressive strength denotes the first local maximum in the stress versus fibre rotation response. The plateau values of σ_c for kinking give the asymptotic strength $\sigma_c = G$, since no local maximum in collapse response is predicted.

where $G^* \equiv [1 + R^2 \tan^2 \beta]G$ and $\gamma_Y^* \equiv \gamma_Y/\sqrt{1 + R^2 \tan^2 \beta}$.

The effect of lock-up upon σ_c becomes significant for $\bar{\phi}_0$ exceeding about 3.5°: then, the local peak in collapse strength is absent and $\sigma_c = G = 100\tau_Y$. When lock-up is ignored, the kinking strength is in close agreement with (4.1) up to a much larger value of initial misalignment $\bar{\phi}_0 \approx 14°$; at larger $\bar{\phi}_0$ values than this critical value, the finite rotation theory displays no local maximum in collapse stress σ_L^∞, and the peak strength is again $\sigma_c = G = 100\tau_Y$.

We can interpret the above observations as follows. For small values of $\bar{\phi}_0$ ($\bar{\phi}_0 \leqslant 3.5°$), the transverse strain e_T within the kink band scales with ϕ^2 whereas the shear strain γ increases linearly with ϕ; consequently, e_T remains negligibly small up to the peak axial stress, and the solution assuming lock-up is in close agreement with the case where lock-up is neglected. In contrast for larger values of $\bar{\phi}_0$, the transverse strain e_T increases linearly with the fibre rotation ϕ, and lock-up occurs immediately; then, the compressive strength in the presence of lock-up diverges from that in the absence of lock-up. It is recognised that the predictions of the kinking analysis are misleading at large values of $\bar{\phi}_0$ since the phenomenon of band broadening at low values of σ_L^∞ is neglected by this approach. For the effects of band broadening to be included, we turn to bending theory.

In order to predict the compressive strength using bending theory, we need to specify the band width in terms of the fibre diameter d. The effect of band width has already been addressed by Fleck et al. (1995); it was found that the compressive strength for any prescribed value of $\bar{\phi}_0$ decreases with increasing d until it attains a plateau value at $w \geqslant 400d$. Observed wavelengths of fibre misalignment in practical unidirectional (and multi-directional) composites are of this order of magnitude (Creighton et al., 2000). Therefore, we shall consider the case of an infinite band of width $w = 400d$, with the band oriented transverse to the fibre direction ($\beta = 0$). Following Fleck and Shu (1995), the distribution of fibre misalignment $\bar{\phi}$ within the infinite band follows a cosine variation with co-ordinate x_1 from the centre of the band, such that

$$\bar{\phi} = \bar{\phi}_0 \cos \frac{\pi}{2}\rho, \quad \text{where } \rho = 2x_1/w \text{ and } |\rho| < 1. \tag{4.2}$$

Elsewhere, the initial fibre misalignment vanishes. For the one dimensional infinite band problem, a finite element mesh of width one element and of length $4000d$ is used, and periodic boundary conditions are applied along the sides of the mesh to enforce the infinite band assumption; the reader is referred to Fleck and Shu (1995) for full details.

The predicted collapse response is shown in *figure 3(b)* for the lock-up case, with $\bar{\phi}_0 = 1°$, $2°$ and $6°$. Here the response is displayed in terms of axial stress σ_L^{∞} versus end shortening u. A local peak in σ_L^{∞} followed by a strong snap-back in load is evident for the cases $\bar{\phi}_0 = 1°$ and $2°$. This behaviour closely resembles that displayed by kinking theory. In contrast, for $\bar{\phi}_0 = 6°$, the applied stress σ_L^{∞} increases monotonically to a plateau value, known as the *steady state band broadening stress*; at this stress level, the microbuckle band broadens with fibre rotation occurring near the boundaries of the broadening band. This phenomenon was first recognised by Moran et al. (1995) and Moran and Shih (1998), and has been analysed recently by Budiansky et al. (1998). Additional in-depth studies on band broadening include those of Vogler and Kyriakides (1999a, 1999b), Hsu et al. (1999) and Jensen (1999). The magnitude of the band broadening stress is of the order of $20\tau_Y$, and is much below the maximum stress $\sigma_c = G = 100\tau_Y$ as predicted by kinking theory for large values of $\bar{\phi}_0$.

The compressive strength σ_c by bending theory is plotted against the amplitude of fibre misalignment $\bar{\phi}_0$ in *figure 4*, for the case $w = 400d$. Results are shown for both cases where lock-up is included and is neglected. As for the case of kinking theory, we note that the peak strength σ_c is sensitive to the misalignment angle $\bar{\phi}_0$, provided $\bar{\phi}_0$ is on the order of a few degrees; in this regime, the effects of lock-up are negligible, and the strength is adequately predicted by (4.1). At larger values of $\bar{\phi}_0$, the strength asymptotes to the band broadening stress level, which is somewhat sensitive to the details of lock-up, but is always much less than the Rosen bifurcation value $\sigma_c = G = 100\tau_Y$. We conclude that the compressive strength is given by (4.1) for small fibre misalignments and by the band broadening stress at large fibre misalignments. The compressive strength switches from the initiation strength, as given approximately by (4.1), to the steady state band broadening strength at a value of $\bar{\phi}_0$ which depends upon the assumed value of band inclination β. It is clear from *figure 4* that the transition value of $\bar{\phi}_0$ is about 3° when lock-up is included and $\beta = 0$. But what about the case $\beta > 0$? The formula (4.1) for the initiation strength remains valid for finite values of β, and we recall the simple formula of Budiansky et al. (1998),

$$\sigma_b = \frac{2}{\sin 2\beta}\tau_Y \tag{4.3}$$

for the steady state band broadening stress σ_b for the case $\beta > 0$. This formula was derived on the basis of negligible transverse stresses within the broadening microbuckle band, and negligible strain hardening, $n = \infty$. The transition value of $\bar{\phi}_0$ at which the strength switches from the initiation value to the band broadening value is obtained by equating (4.1) and (4.3) for the case $n = \infty$, giving

$$\bar{\phi}_0 \approx \frac{\sin 2\beta}{2} - \gamma_Y. \tag{4.4}$$

In order to assess the significance of β upon the transition waviness value, let us assume that β is in the range 15° to 30°. Then, the transition value of $\bar{\phi}_0$ is in the range 14° to 24° (assuming $\gamma_Y = 0.01$). We conclude that the compressive strength is given by (4.1) for practical composites provided $\bar{\phi}_0$ is less than about 14°; this is the case for composites processed via the pre-preg route, but is not necessarily the case for woven laminates.

4.2. Microbuckle initiation from a finite wavy patch

Practical composites contain misaligned fibres within finite patches, rather than the mathematical idealisation of an infinite band of waviness. In this subsection we report the predicted compressive strength of a unidirectional lamina containing a circular region of waviness, of diameter D in the range of $20d \leqslant D \leqslant 400d$. Previous studies of this type have been limited to small values of fibre misalignment, on the order of a few degrees. Here, we explore the effect of large fibre misalignments upon strength. The distribution of $\bar{\phi}$ within the wavy patch is taken as

$$\bar{\phi} = \bar{\phi}_0 \cos \frac{\pi}{2} \rho, \quad \rho < 1, \tag{4.5}$$

where ρ is defined by

$$\rho = 2\sqrt{x^2 + y^2} \Big/ D \tag{4.6}$$

in terms of the local Cartesian axes (x, y) at the mid-point of the lamina, as shown in *figure 5(a)*. Predictions are made using the bending theory of section 3, with a square mesh of side length $100D$. The material properties have already been stated at the end of section 3; unless otherwise stated, deformation theory is used and the effect of volumetric lock-up is included in the analysis. The predicted compressive strength is plotted as a function of $\bar{\phi}_0$ in *figure 5(a)* for selected values of D/d, and $n = 3$ and 10. The strength predictions for $D = 400d$ and $n = 3$ are compared in *figure 5(b)* with the infinite band results for band inclinations $\beta = 0°$ and 30°, with $w = 400d$ and $n = 3$; again, lock-up is assumed and the deformation theory version of plasticity is adopted. We can make the following deductions from *figures 5(a)* and *5(b)*, taken together.

(i) The compressive strength for both the circular patch of waviness and the infinite band decreases with increasing $\bar{\phi}_0$, for $\bar{\phi}_0$ less than about 10°. At larger values of initial misalignment, the strength asymptotes to a constant value.

(ii) The strength for a circular patch of waviness increases with decreasing n and with decreasing diameter D of patch. Similar findings have already been reported for collapse from an infinite band and from circular patches of small initial waviness, see Fleck et al. (1995) and Shu and Fleck (1997). The sensitivity of strength to the strain hardening index n is greatest at large values of $\bar{\phi}_0$: this is supported by the observation that the initiation strength as given by (4.1) for small $\bar{\phi}_0$ is relatively insensitive to n, whereas the band broadening stress increases strongly with decreasing n, as discussed by Budiansky et al. (1998).

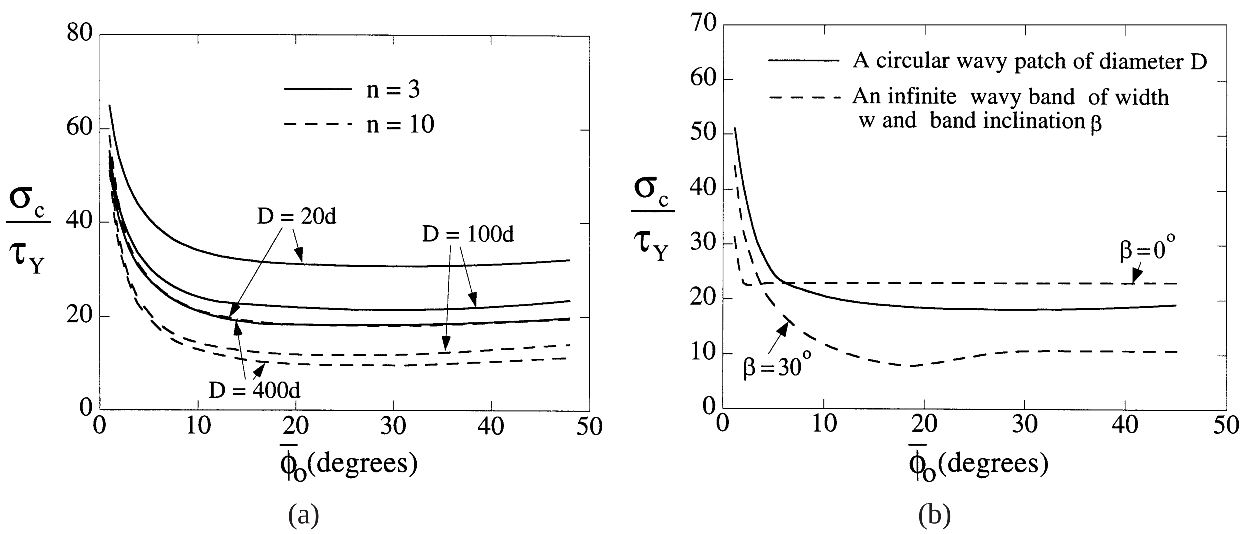


Figure 5. (a) Compressive strength versus amplitude of fibre misalignment $\bar{\phi}_0$ for a circular patch of fibre misalignment, for selected values of diameter D and strain hardening index n. (b) Comparison of compressive strength for a circular imperfection of diameter $D = 400d$ and an infinite band of width $w = 400d$, for $n = 3$. In both figures, bending theory with lock-up is assumed, and the deformation theory version of plasticity is adopted.

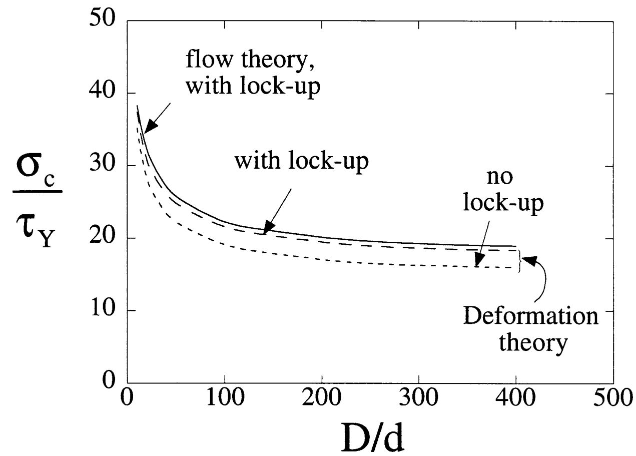


Figure 6. Effect of diameter D of circular wavy patch upon compressive strength, according to bending theory for an amplitude $\bar{\phi}_0 = 30°$ and $n = 3$. The effect of lock-up and the choice of constitutive relation (deformation theory versus flow theory) is explored.

(iii) The compressive strength due to a circular patch of waviness lies between the strengths for an infinite band at $\beta = 0°$ and 30°, for $\bar{\phi}_0$ greater than about 5°. This is consistent with our observation that, at peak load, microbuckling from a circular patch of waviness is activated mostly along an orientation of $\beta \approx 10°$; contours of fibre rotation are omitted for the sake of brevity.

The sensitivity of the compressive strength to the wavy patch size, to the lock-up assumption and to the choice of a constitutive law (deformation theory versus flow theory) is shown in *figure 6* for the case of a circular patch of large misalignment, $\bar{\phi}_0 = 30°$. A clear size effect is evident, with the compressive strength doubling as the imperfection size D is reduced from $400d$ to $20d$. As D/d is reduced to zero, the strength increases to the elastic bifurcation strength $\sigma_c = G$, while the strength levels off to a constant value of $\sigma_c \approx 20\tau_Y$ at large D/d values on the order of 400. Relaxation of the lock-up assumption, or the use of flow theory rather than deformation theory have only minor effects upon the compressive strength.

5. Compressive strength of a composite beam in bending

Composite structures such as beams are often subjected to bending. Bend tests are also used to measure the stiffness and strength of composites. Thus, it is important to determine whether the size effect on compressive strength is present in bending in addition to uniaxial compression. Here, we consider the highly idealised case of a unidirectional composite beam, of height H, containing a transverse band of waviness, of width w; the fibre misalignment $\bar{\phi}$ within the infinite band follows the cosine variation as defined by (4.2). The deformation theory version of bending theory of section 3 is used to predict the compressive strength σ_b on the outermost fibre of the beam under pure bending, and the compressive strength σ_c under uniaxial compression.

Predictions are shown in *figure 7* for the choice $w = 20d$, $n = 3$, $\bar{\phi}_0 = 2°, 4°, 30°$ and with lock-up included in the analysis. There is a dramatic difference in the severity of the size effect for bending and for uniaxial compression: in uniaxial compression the free surface has a minor influence on the compressive strength and σ_c is almost independent of the H/d ratio (as discussed previously by Fleck et al. (1998)). In contrast, the bending strength σ_b increases by a factor of about three when H/d is reduced from 1000 to 20; this may be interpreted as follows. In the bending case, the outermost fibres of the beam are supported by adjacent stiff material, and so microbuckling of the surface layers is delayed. The effect is most pronounced when the stress gradient across the height of the beam is large, which is the case for thin beams. As the beam height H

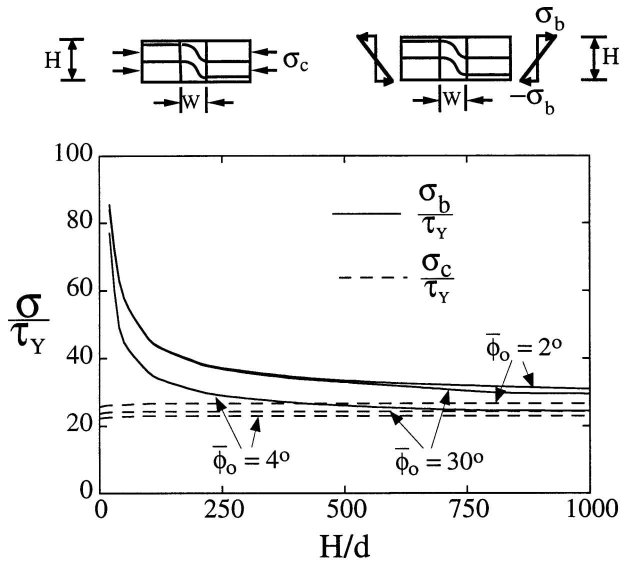


Figure 7. Effect of beam height H on the bending strength σ_b and the uniaxial compressive strength σ_c for a parallel-sided band of fibre waviness of width $20d$ and amplitude $\bar{\phi}_0 = 2^\circ$, 4° and 30°. The deformation theory version of bending theory is employed, with $n = 3$ and lock-up included.

increases, the maximum compressive stress under bending σ_b approaches the uniaxial compressive strength σ_c. The difference in strengths for a beam in bending and a beam under direct compression is significant for beams of height H less than about $1000d$. Typically, the diameter of carbon fibres is about $d = 5\ \mu$m, and so we expect size effects to be present in beams of height up to 5 mm. Further experiments along the lines of Wisnom (1992, 1994) would be welcomed to quantify the effect fully.

6. Concluding remarks

We conclude from the results of this study that a one-dimensional infinite band analysis in the form of kinking theory is adequate for predicting the microbuckling strength of a fibre composite, provided the region of fibre misalignment is large in relation to the fibre diameter, and provided the amplitude of the misalignment is on the order of a few degrees. For larger values of initial waviness, a local maximum in peak strength is not attained; instead, the effect of fibre bending is to lead to band broadening at constant applied stress, and bending theory is needed in order to predict the compressive strength. Fibre bending theory is also useful for characterising the effect of the initial width w of the band of misaligned fibres upon the compressive strength. Narrow imperfections are less deleterious than wide imperfections due to the stabilising effects of fibre bending.

The predicted compressive strength for a circular patch of waviness has been obtained as a function of misalignment amplitude $\bar{\phi}_0$ and imperfection size using bending theory. Provided $\bar{\phi}_0$ exceeds about 10°, the compressive strength σ_c is in the range 10–$30\tau_Y$, depending upon the strain hardening exponent, and upon the diameter D of imperfection. Typical polymer matrix composites have a shear yield strength τ_Y of about 50 MPa, and so the predicted compressive strength is in the range 500–1500 MPa for a wide range of imperfection geometry. Observed strength values for unidirectional and woven composites lie in this range, as reviewed by Fleck (1997).

Finally, this study supports the findings of Wisnom (1992, 1994) that the bend strength of composites is particularly sensitive to the magnitude of the imposed stress-gradient: the bend strength of the outermost fibre of the beam increases with diminishing height H of the beam. In contrast, the uniaxial compressive strength of a composite panel is hardly influenced by the panel width.

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