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FOAM TOPOLOGY BENDING VERSUS STRETCHING DOMINATED ARCHITECTURES

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Abstract—Cellular solids can deform by either the bending or stretching of the cell walls. While most cellular solids are bending-dominated, those that are stretching-dominated are much more weight-efficient for structural applications. In this study we have investigated the topological criteria that dictate the deformation mechanism of a cellular solid by analysing the rigidity (or otherwise) of pin-jointed frameworks comprising inextensional struts. We show that the minimum node connectivity for a special class of lattice structured materials to be stretching-dominated is 6 for 2D foams and 12 for 3D foams. Similarly, sandwich plates comprising of truss cores faced with planar trusses require a minimum node connectivity of 9 to undergo stretching-dominated deformation for all loading states.

Keywords: Foams; Mechanical properties; Structure

1. INTRODUCTION

Over the past few years, a variety of metallic and polymeric foams have been produced with the aim of developing lightweight structures which are adequately stiff and strong. Numerous studies on open-cell foams have shown that both the stiffness and strength of the foams are governed by cell wall bending for all loading conditions; the first scales as $\bar{\rho}^2$ and the second as $\bar{\rho}^{1.5}$, where $\bar{\rho}$ is the relative density of the foam; see Gibson and Ashby [1], and Ashby *et al*. [2]. Most closed-cell foams also follow these scaling laws, as the cell faces, which carry membrane stresses, buckle or rupture at stresses so low that their contribution to stiffness and strength is small, leaving the cell edges to carry most of the load.

An open-cell foam can be treated as a connected set of pin-jointed struts by the following argument. Consider the pin-jointed frames shown in Fig. 1. The frame in Fig. 1(a) is a *mechanism*. When loaded, the struts rotate about the joints and the frame collapses; it has neither stiffness nor strength. The triangulated frame shown in Fig. 1(b) is a *structure*: when loaded the struts support axial loads, tensile in some, compressive in others. Thus, the deformation is *stretching-dominated* and the frame collapses by stretching of the struts. Imagine now that the joints of both

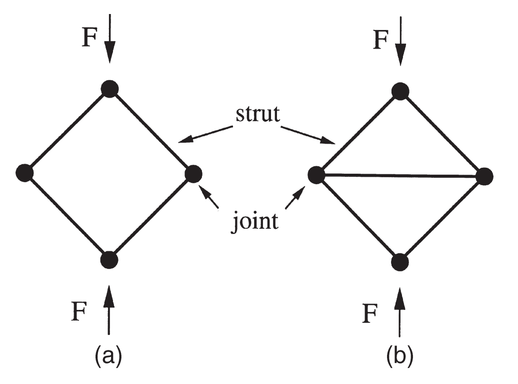

Fig. 1. (a) A mechanism; (b) a structure.

frames are frozen to prevent free rotation of the struts. On loading the first frame, the struts can no longer rotate. The applied load induces bending moments at the frozen joints, and these cause the struts to bend. This is the situation in most foam structures. However, freezing the joints of the triangulated structure has virtually no effect on the macroscopic stiffness or strength; although the struts bend, the frame is still stretching-dominated and the collapse load is dictated mainly by the axial strength of the struts.

Foams that are stretching-dominated are more efficient from a weight standpoint; for example, a stretching-dominated foam is expected to be about ten times as stiff and about three times as strong as a bending-dominated foam for a relative density $\bar{\rho} = 0.1$. In this paper we discuss the topological criteria

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that dictate whether a foam will be bending or stretching dominated with the overall aim of developing an understanding of the microstructure that maximises the strength (or stiffness) to weight ratio of a nearly isotropic foam.

2. RIGIDITY OF FRAMES

To establish the topological criteria for stretch-dominated behaviour, we proceed to analyse the rigidity (or otherwise) of an assembly of inextensional pin-jointed struts.

2.1. Maxwell's criterion

Maxwell [3] suggested an algebraic rule setting out the condition for a pin-jointed frame of b struts and j frictionless joints to be both statically and kinematically determinate ie. to just be rigid. In 2 and 3 dimensions the criteria are:

$$b = 2j-3 \text{ and} \quad (1a)$$

$$b = 3j-6 \quad (1b)$$

respectively. These criteria are necessary conditions for rigidity, but not in general sufficient conditions as they do not account for the possibility of states of self stress (a self-equilibrated state of strut tensions in the absence of external load) and of mechanisms. A generalisation of the Maxwell rule in 3D is given by [4]:

$$b-3j+6 = s-m \quad (2)$$

where s and m count the states of self stress and mechanisms, respectively, and each can be determined by finding the rank of the equilibrium matrix that describes the frame in a full structural analysis [5]. A just rigid framework (i.e. a framework that is both statically and kinematically determinate) has $s = m = 0$. The nature of Maxwell's rule as a necessary rather than sufficient condition is made clear by examination of equation (2): vanishing of the LHS in equation (2) only implies that the number of mechanisms and states of self-stress are equal, not that each equals zero.

Having reviewed the main concepts on the rigidity of frameworks we now proceed to analyse the rigidity criteria of infinite frameworks representing foam microstructures. Consider a large pin-jointed framework with j joints and an average connectivity (number of struts at a node) Z. The total number of struts b in the framework is $\approx jZ/2$. Thus, by Maxwell's rule the necessary (but not sufficient) condition for rigidity is $Z = 4$ in the 2D case and $Z = 6$ in the 3D case.

The infinite 2D cubic framework sketched in Fig. 2(a) has $Z = 4$ and admits one mechanism and one state of self stress. Thus, while the framework satisfies the Maxwell criterion it is not rigid. On the other hand, an infinite cubic framework with edge cells braced as shown in Fig. 2(b) also has an average connectivity $Z = 4$ and represents a rigid framework that satisfies the Maxwell criterion. Similarly, a 3D rigid framework that satisfies the Maxwell criterion is a cubic framework with braced cells along three orthogonal edge planes.

2.2. Periodic frameworks

Open-cell foam microstructures can be treated as periodic frameworks. Here we identify classes of unit cells from which rigid periodic structures can be synthesised. Evidently, periodic frameworks constructed from rigid unit cells will necessarily be rigid. The converse may not hold: frameworks constructed from non-rigid unit cells may or may not be rigid. For example, the unit cell of the framework in Fig. 3 is not rigid while the framework synthesised from this cell is rigid. This is because the rigid braced square sub-cells link up to form a rigid skeleton as depicted in Fig. 3.

Examples of some idealised cell shapes are shown in Fig. 4. Isolated cells that satisfy Maxwell's criterion and are rigid are labelled "YES" while "NO" means the Maxwell condition is not satisfied and that the cell is a mechanism. It is worth mentioning here that any convex simply-closed polyhedron with triangular faces satisfies the Maxwell criterion and is rigid (see Appendix 9 in Calladine). It is generally assumed that the best model for a cell in a foam approximates a space filling shape. However, none of the space filling shapes (indicated by numbers 2, 3, 4, 6, 7 and 8) are rigid. In fact, we could not identify any rigid space filling cell in the 3D case and only succeeded in synthesising rigid periodic 3D frameworks from combinations of rigid cells (e.g. the tetrahedron and octahedron in combination fill space to form a rigid framework).

The point has been made that stretch-dominated structures offer greater stiffness and strength per unit weight than those in which the dominant mode of deformation is by bending. It should be realised that *minimal* stretch-domination offers only marginal gain; for the full gain to be realised, the structure must be *predominantly* stretch-dominated. The point is brought out by Fig. 2(c) which shows a two-dimensional "material" made by repeating a unit cell with a geometry like that of Fig. 2(b). On a length-scale large compared with the size of the unit cell, the structure is homogeneous and it cannot be deformed without stretch-deformation of the struts that form the shaded square grid of sub-cells. However, the full gain possible by stretch-domination is not realised in this case because the interior of each square still deforms by bending of cell edges. The structure resembles that of a composite made up of a grid of stiff material containing regularly dispersed soft "inclusions". The upper bound for the modulus is then given by a rule of mixtures (or, more rigorously, by the H–S [7] upper bound, which is lower); and as the

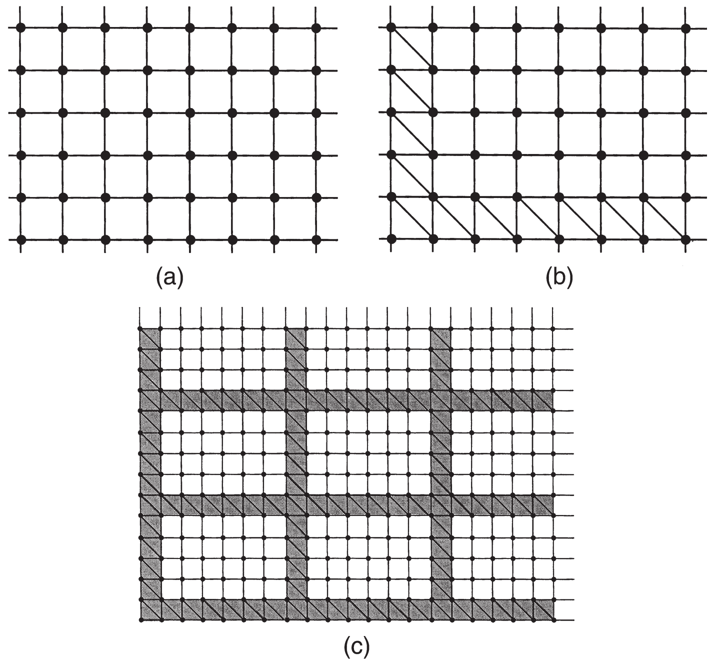

Fig. 2. Three types of frameworks, (a) a mechanism that satisfies Maxwell's criterion (b) a statically determinate structure that satisfies Maxwell's criterion (c) A statically indeterminate structure that is "minimal" stretch-dominated.

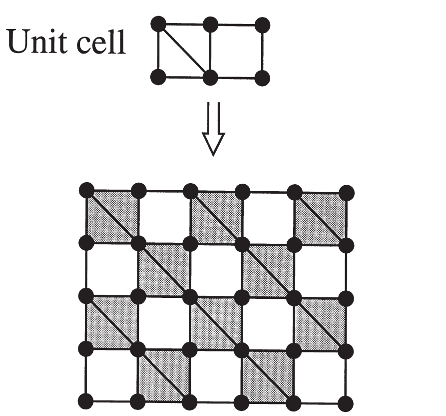

Fig. 3. Example of a non-rigid unit cell that can be stacked to form a rigid framework.

area fraction of this stiffer phase is reduced, the stiffness falls to that of the less stiff material.

The same considerations hold in three dimensions. If a space-filling unit cell is made by assembling stretch and bending dominated sub-units, and these are stacked (as in Figs 2 and 3) to retain some stretch domination, the overall stiffness may still lie well below the "ideal", meaning the best achievable. It is therefore relevant to examine the optimisation of 3D lattice structures to maximise stretch-dominated behaviour. This we explore in the next section.

2.3. A special class of periodic frameworks

In this section we analyse a special class of frameworks with nodes which are all *similarly situated*; nodes are said to be similarly situated if the rest of the framework appears the same and in the same orientation if viewed from any of the nodes [8]. We now proceed to determine the minimum connectivity Z for frameworks of this class to be rigid.

Consider an infinite framework of inextensional struts with similarly situated nodes and a connectivity Z. Without any loss of generality all the struts are assumed to be of unit length. We define a Cartesian co-ordinate system x_i with it's origin at a node O. For an applied macroscopic strain field E_{ij}, the displacement $u_i^{(k)}$ of the ends of each strut (k) connected to O is given by

$$u_i^{(k)} = E_{ij} n_j^{(k)};\ k = 1,2\ldots Z \tag{3}$$

where $\mathbf{n}(^k)$ is a unit vector in the direction of strut k. However, as the struts are inextensional the displacements equation (3) must satisfy each of the constraints

$$u_i^{(k)} n_i^{(k)} = 0;\ k = 1,2\ldots Z. \tag{4}$$

The framework under consideration has similarly situated nodes. Thus, each strut shares a node with a collinear partner, which implies that equation (4)

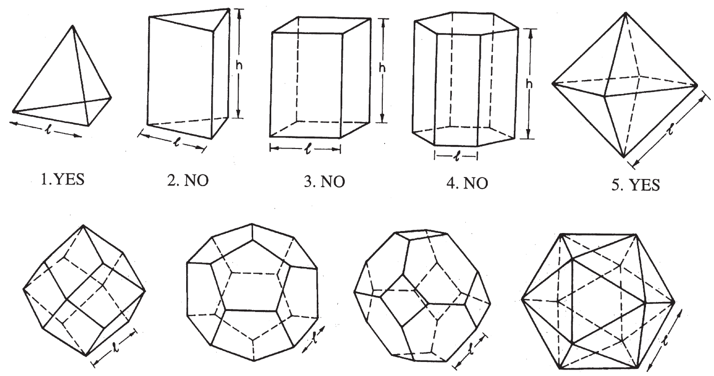

Fig. 4. Three-dimensional polyhedral cells that do, or do not, satisfy the Maxwell criterion.

represents only $Z/2$ *independent* equations. Combining the $Z/2$ independent equations in equation (4) with the corresponding equations in equation (3) we get

$$E_{ij}n_j^{(k)}n_i^{(k)} = 0; \; k = 1,2\ldots\frac{Z}{2}, \tag{5}$$

where $\mathbf{n}^{(k)}$ are now all non-collinear. We proceed by writing equation (5) in the matrix form:

$$\mathbf{NE} = 0 \tag{6}$$

where

$$\mathbf{N} = \begin{pmatrix} n_1^{(1)^2} & n_1^{(1)}n_2^{(1)} & n_2^{(1)^2} \\ n_1^{(2)^2} & n_1^{(2)}n_2^{(2)} & n_2^{(2)^2} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ n_1^{(Z/2)^2} & n_1^{(Z/2)}n_2^{Z/2} & n_2^{(Z/2)^2} \end{pmatrix} \text{and } \mathbf{E} = \begin{pmatrix} E_{11} \\ 2E_{12} \\ E_{22} \end{pmatrix} \tag{7}$$

in the 2D case. Similarly, in the 3D case $\mathbf{N}$ is a $Z/2 \times 6$ matrix formed from the unit vectors $\mathbf{n}$ while $\mathbf{E}$ is a 6×1 matrix of the 3D strain components. The necessary and sufficient condition for rigidity is that the only solution to the homogeneous system of linear equations equation (6) is $E_{ij} = 0$ i.e. rank($\mathbf{N}$) = 3 and 6 in the 2D and 3D cases, respectively. Using the observation that $\mathbf{N}$ is formed from non-collinear vectors it can be easily shown that

$$\text{rank}(\mathbf{N}) = \begin{cases} \frac{Z}{2} \leq 3 \text{ for the 2D case,} \\ \frac{Z}{2} \leq 6 \text{ for the 3D case.} \end{cases} \tag{8}$$

Thus, the necessary and sufficient condition for rigidity of 2D and 3D frameworks is that the connectivity $Z = 6$ and $Z = 12$, respectively. Note that if Z exceeds these values the framework is redundant from a rigidity standpoint. On the other hand, if for example a 3D framework has $Z = c < 12$, then the solutions to equation (6) form a vector space of dimension $(6-c/2)$ corresponding to $(6-c/2)$ independent mechanisms. Thus, a cubic framework with $Z = 6$ has 3 independent mechanisms.

Examples of frameworks that satisfy the above rigidity conditions are a 2D fully triangulated framework and a 3D "face-centred cubic" lattice structure (see Deshpande *et al*. [9] for a detailed analysis of the "Octet-truss" lattice material with a FCC microstructure). It is worth mentioning here that in 2D, a triangle is the only rigid polygon that can be constructed from inextensional pin-jointed struts. Thus, a rigid 2D framework with equal connectivity at all nodes is necessarily fully triangulated.

2.3.1. Comparison with optimal microstructures. Isotropic cellular materials that attain the H–S [7] upper bounds for the bulk and shear moduli of a voided solid maximise the stiffness to weight ratio. A number of classes of two phase composites attain these bounds. For example, Norris [10] and Milton [11] proposed differential schemes for constructing composite structures with the extremal H–S bulk and shear moduli. While Milton [11] used a laminate microstructure, Norris [10] employed a coated sphere architecture. However, the procedures suggested by both these authors are incremental and require an infinite number of mixing processes. Moreover, the procedures do not specifically describe the underlying microstructure of the composite. On the other hand, Francfort and Murat [12] suggested the so called "rank" laminates which attain both the bulk and shear H–S bounds with a finite number of layering directions. Rank laminates are obtained by a sequential process where at each stage the previous laminate is laminated again with a single phase (always the same) in a new direction. Thus, a rank-n laminate is pro-

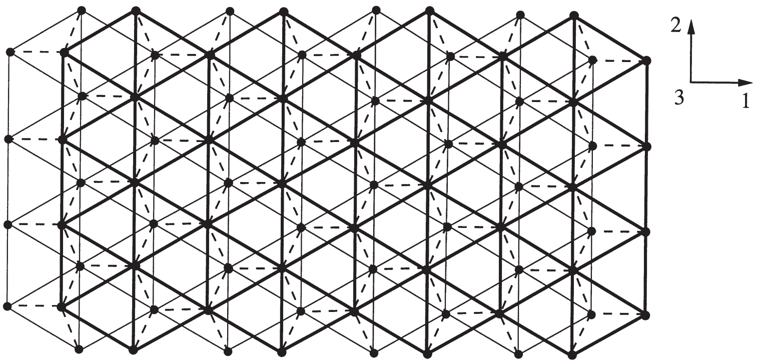

Fig. 5. Schematic drawing of sandwich plate with a truss core faced with planar trusses. In this example the sandwich plate comprises of a tetrahedral truss core faces with a planar fully triangulated truss. The darker lines are top face members, the lighter lines are the bottom face members, and the dashed lines are the core members.

duced by *n* such successive laminations. Francfort and Murat [12] showed that while in the 2D case, isotropic rank-3 laminates have the extremal bulk and shear moduli, in the 3D case rank-6 laminates are the optimal microstructures. Thus, there exist a variety of multi length-scale microstructures with extremal values of the bulk and shear moduli. However, no single length-scale microstructure that attains both the bulk and shear H–S bounds has been proposed to-date. It is conjectured that the FCC microstructure designed using the rigidity prescription detailed above is close to the optimal single length-scale cellular microstructure.

3. RIGIDITY OF TRUSS PLATES

Motivated by recent advances in manufacturing technology, attention has been focussed on minimum weight sandwich plates comprised of truss cores faced with planar frameworks (see for example Wicks and Hutchinson [13]). A schematic drawing of one such sandwich plate comprising of a tetrahedral truss core faced with fully triangulated planar trusses is shown in Fig. 5. As in the case of foams, from a weight standpoint it is advantageous if the deformation of these sandwich truss plates is stretching-dominated. Hence, we discuss the criteria for periodic sandwich truss plates with equal connectivity at all nodes to be rigid in the sense explained above.

Sandwich construction exploits a low-density core to separate thin face sheets in order to attain a high cross-sectional bending stiffness. It is thus reasonable to assume that the deformation of the faces should be stretching-dominated for all applied in-plane strains (E_{11},E_{12},E_{22}). Thus, the planar trusses forming the face sheets are necessarily fully triangulated 2D frameworks with $Z = 6$ (see Section 2.3). It now remains to determine the minimum number of core struts at each node of the face sheets so as to ensure rigidity of the truss plate.

The rigid face sheets ensure that the truss plate is rigid with respect to in-plane straining i.e. $E_{11} = E_{12} = E_{22} = 0$. Thus, the core only needs to constrain the strains (E_{13},E_{23},E_{33}). An analysis similar to that detailed in Section 2.3 shows that the necessary and sufficient condition for strains (E_{13},E_{23},E_{33}) to be constrained to equal zero is that three non-coplanar core struts are connected to every node. Thus, the minimum connectivity of a periodic rigid sandwich truss plate is $Z = 6 + 3 = 9$. A sandwich truss plate comprising triangulated face sheets and a tetrahedral core Fig. 5 satisfies this criterion and has stretching-dominated deformation for all loading states.

4. DESIGN IMPLICATIONS

The modulus and *initial* yield strength of a stretching-dominated cellular solid are much greater than those of a bending-dominated cellular material of the same relative density. This makes stretching-dominated cellular solids attractive alternatives to bending-dominated foamed materials for lightweight structural applications. However, unlike bending dominated foams, in compression the stretching-dominated materials have a softening post-yield response due to the buckling of the struts [9]. Thus, these materials may be less attractive as energy-absorbers since this application requires a stress–strain response with a long, flat plateau.

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