Bounds and estimates for the effect of strain gradients upon the effective plastic properties of an isotropic two phase composite

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Abstract

Predictions are made for the size effect on strength of a random, isotropic two phase composite. Each phase is treated as an isotropic, elastic-plastic solid, with a response described by a modified deformation theory version of the Fleck-Hutchinson strain gradient plasticity formulation (Fleck N A and Hutchinson J W, 2001, *J. Mech. Phys. Solids*, **49**, 2245-2272.) The essential feature of the new theory is that the plastic strain tensor is treated as a primary unknown on the same footing as the displacement. Minimum principles for the energy and for the complementary energy are stated for a composite, and these lead directly to elementary bounds analogous to those of Reuss and Voigt. For the case of a linear hardening solid, Hashin-Shtrikman bounds and self-consistent estimates are derived. A nonlinear variational principle is constructed by generalizing that of Ponte Castañeda (1992, *J. Mech. Phys. Solids*, **40**, 1757-1788). The minimum principle is used to derive an upper bound, a lower estimate and a self-consistent estimate for the overall plastic response of a statistically homogeneous and isotropic strain gradient composite. Sample numerical calculations are performed to explore the dependence of the macroscopic uniaxial response upon the size scale of the microstructure, and upon the relative volume fraction of the two phases.

1. Introduction

In recent years there has been increasing interest in the observation and prediction of size effects in metal plasticity. Size effects can be associated with various length scales, corresponding to different physical mechanisms. For example, in age hardened aluminium alloys, precipitate spacing is on the order of 1-100 nm and the main strengthening mechanism is the interaction of individual dislocations with precipitates. Orowan strengthening falls within this regime: the elevation in strength due to precipitates scales with the inverse spacing of neighbouring precipitates. But at larger length scales, on the order of 0.1-10 μm, dislocations display collective behaviour, and macroscopic strengthening is dependent upon the density of dislocations. The classical papers by Nye (1953) and Ashby (1970, 1971) argue that plastic strain gradients induce *geometrically necessary dislocations* and the interaction of these dislocations with *statistically stored dislocations* (accumulated by strain and not strain gradient) leads to a size effect. Observed size effects at this *meso-level length scale* include an increasing indentation hardness with diminishing indent size (Stelmashenko *et al.*, 1993; Poole *et al.*, 1996; Nix and Gao, 1998), an elevated strength of SiC particulate reinforced aluminium alloys with diminishing diameter of particle at fixed volume fraction (Lloyd, 1994), and an increasing shear stress versus shear strain curve with diminishing radius in a series of experiments on the torsion of thin copper wires (Fleck *et al.*, 1994). Size effects also emerge in the localisation of deformation – an internal material length sets the width of a shear band (Aifantis, 1984; Mikkelsen, 1999; Sluys and Estrin, 2000).

A number of strain gradient plasticity models have been developed to capture phenomenologically the size effect at the meso-level (see, for example, Aifantis, 1984; Fleck and Hutchinson, 1997, 2001; Gurtin, 2000), and evidence is accruing that a higher order theory with additional boundary conditions is required to produce the observed size effects. For example, Shu *et al.* (2001) and Bittencourt *et al.* (2003) have used discrete dislocation methods to calculate the shear strain distribution across the layer thickness of a thin metallic strip sandwiched between two substrates. They found that a distinct boundary layer develops, and they demonstrated that a strain gradient theory with higher order boundary conditions is necessary in order to reproduce the boundary layer and the associated dependence of shear strength upon layer thickness.

A limited number of attempts have been made to predict the size effect in two phase composites and the Hall-Petch relation in polycrystals. Smyshlyaev and Fleck (1994, 1995, 1996) have performed homogenisation calculations of this type by employing the deformation theory version of the Fleck and Hutchinson (1993, 1997) strain gradient theory. In this theory, the strain energy is a non-linear function of both the strain ε_{ij}, derived from a continuous displacement field $u_i(x_j)$, and the first spatial gradient of strain, $\varepsilon_{ij,k}$. The theory fits within the Toupin (1962) and Mindlin (1964, 1965) variational framework, and a third order tensor τ_{ijk} emerges as the work-conjugate to the strain gradient $\varepsilon_{ij,k}$. Higher order boundary conditions arise naturally and the strain gradient theory is able to reproduce a number of observed size effects, as discussed by Fleck and Hutchinson (1997). However, the flow theory version of this gradient theory has a few unsatisfactory features:

(i) purely elastic straining leads to the generation of both the symmetric Cauchy stress σ_{ij} (work-conjugate to ε_{ij}) and the higher order stress τ_{ijk}. Consequently, the yield surface is stated in terms of the Cauchy stress and the higher order stress τ_{ijk}. The physical basis of a higher order stress in the absence of dislocations is on a much smaller length scale than the micron-scale plasticity envisaged here.

(ii) The strain ε_{ij} is decomposed additively into elastic and plastic parts, with the elastic part related to the Cauchy stress via a fourth order elastic compliance tensor. Similarly, the total strain gradient $\varepsilon_{ij,k}$ is decomposed into an elastic and a plastic part, with the elastic part related to τ_{ijk} via a sixth order elastic compliance tensor. Whilst the total strain gradient $\varepsilon_{ij,k}$ is necessarily the spatial gradient of the total strain ε_{ij}, the plastic part of $\varepsilon_{ij,k}$ is not the spatial derivative of the plastic part of ε_{ij}. This leads to a theory which is mathematically correct but subtle in interpretation.

Recently, Fleck and Hutchinson (2001) have reformulated their strain gradient plasticity theory to overcome the above two deficiencies. Both deformation and flow theory versions have been derived, and the reformulation is based upon treating the displacement field $u_i(x_j)$ and the effective plastic strain $\varepsilon_P(x_j)$ as independent kinematic quantities to be determined by the governing field equations. For the non-linear deformation theory version, the energy is partitioned into an elastic contribution dependent upon the elastic strain, and upon a plastic contribution which is dependent upon both $\varepsilon_P\left(x_j\right)$ and its spatial gradient $\nabla\varepsilon_P\left(x_j\right)$. Higher order stress emerges as the work conjugate of $\nabla\varepsilon_P\left(x_j\right)$, with associated higher order boundary conditions.

Scope of Study

The present study makes use of the recent Fleck and Hutchinson (2001) strain gradient theory in order to predict the size effect for a two phase, statistically isotropic composite under uniform macroscopic straining. The two phases of the composite are perfectly bonded, and each is described by a non-linear deformation theory version of plasticity, with strain gradient effects included at the local level. The admission of strain gradient plasticity at the local level provides enhancement of the macroscopic flow stress via local gradients, and introduces a scale effect. The phenomenological constitutive law adopted is a modification of that introduced by Fleck and Hutchinson (2001), and allows for easier implementation. The study begins with an outline of the non-linear constitutive law employed for each phase, and the development of a Hashin-Shtrikman type variation principle for the *linear* strain gradient composite. Selected results for Hashin-Shtrikman type bounds and self-consistent estimates are given for the linear case. Then, an upper bound for the *non-linear* strain gradient composite is derived by extending the variational formulation of Ponte Castañeda (1991, 1992) to include the role of strain gradients. A lower estimate and a self-consistent estimate are also obtained. For comparison purposes, the elementary Voigt and Reuss bounds are included. The variational method is used to explore the sensitivity of macroscopic strength of a two-phase non-linear composite to the size of each phase in addition to the volume fraction and relative strength of each phase.

2. A strain gradient constitutive law for an isotropic, elastic-plastic deformation theory solid

Fleck and Hutchinson (2001) have introduced a strain gradient version of isotropic deformation theory, employing the von Mises plastic strain ε_P and its spatial gradient $\nabla\varepsilon_P$ as variables. First, we outline this theory, and second we modify it to a form which allows for the energy statement to be written more directly as a function of total strain ε_{ij} and of the internal state variables $\left(\varepsilon_P, \nabla\varepsilon_P\right)$ without the involvement of stress.

In the Fleck and Hutchinson (2001) gradient deformation theory, the plastic strain tensor is written as

$$\varepsilon_{ij}^P = \varepsilon_P m_{ij} \;, \qquad \varepsilon_P \geq 0 \tag{2.1}$$

where the scalar ε_P is a free variable (on the same footing as the displacement field u_i), and the direction m_{ij} of plastic strain is co-directional with the stress deviator σ'_{ij} such that

$$m_{ij} = \frac{3}{2}\frac{\sigma'_{ij}}{\sigma_e} \tag{2.2}$$

Here, σ_e is the usual von Mises stress measure, as expressed by $\sigma_e = \sqrt{\tfrac{3}{2}\sigma'_{ij}\,\sigma'_{ij}}$. Note that m_{ij} is homogeneous of degree zero in σ'_{ij}, with $\left|m_{ij}\right| = \sqrt{3/2}$. Combination of (2.1) with (2.2) demonstrates that ε_P is the von Mises plastic strain, such that $\varepsilon_P = \sqrt{\tfrac{2}{3}\varepsilon_{ij}^P\varepsilon_{ij}^P}$. An overall effective plastic strain measure E_P is introduced, to give a combined measure of hardening due to strain, as characterised by ε_P, and strain gradients, as characterised by $\varepsilon_{P,i}$:

$$E_P \equiv \left[\,\varepsilon_P{}^2 + \lambda^2\varepsilon_{P,i}\varepsilon_{P,i}\,\right]^{1/2} \tag{2.3}$$

The underlying physical argument behind the relation (2.3) is that E_P provides an overall scalar measure of dislocation content, with ε_P giving a measure of the density of statistically stored dislocations and $\left|\varepsilon_{P,i}\right|$ providing a measure of the density of geometrically necessary dislocations, associated with strain gradients, see for example Fleck *et al.* (1994). Fleck and Hutchinson (2001) also consider a more sophisticated version of strain gradient theory, wherein E_P is expressed as a function of the three quadratic invariants of the plastic strain gradient $\varepsilon_{ij,k}^P$; these three invariants dictate the existence of three material length scales $\left(\lambda_1, \lambda_2, \lambda_3\right)$ and not simply λ. Fleck and Hutchinson argue that it is necessary to consider all three length scales in order to predict the magnitude of the size effect in a wide range of phenomena. In the present study, the focus is on the prediction of size effects for a composite under macroscopic uniaxial straining, and the main features will emerge by limiting attention to a single length scale.

Fleck and Hutchinson (2001) defined a potential energy functional for a body occupying a domain Ω as

$$\Phi\left(u_i,\varepsilon_P\right)=\int_\Omega\left\{\frac{1}{2}L_{ijkl}\left(\varepsilon_{ij}-\varepsilon_P m_{ij}\right)\left(\varepsilon_{kl}-\varepsilon_P m_{kl}\right)+\int_0^{E_P}f\left(e\right)de\right\}dx-\int_{S_T}\left(T_i^0 u_i+t^0\varepsilon_p\right)dS \quad (2.4)$$

where L_{ijkl} is the elastic stiffness tensor and $f(e)$ denotes the uniaxial tensile stress versus plastic strain curve of the material. (The present development considers small displacements and infinitesimal straining, so the distinctions between the various stress and strain measures associated with finite deformations are ignored.) On the portion S_T of the surface the stress traction T_i^0 and the scalar higher order traction t^0 are prescribed; u_i and ε_P have prescribed values on the complementary portion S_U. Then, the actual solution minimises $\Phi(u_i,\varepsilon_P)$ provided L_{ijkl} is positive definite and $h\equiv df/d\varepsilon_P$ is positive. The first contribution to the volume integral is the elastic energy density while the second is the plastic work density. In classical deformation theory the plastic work density is $\int_0^{\varepsilon_P}f(e)de$. Here, the plastic work density is evaluated at E_P rather than ε_P.

Note that the expression (2.4) for $\Phi(u_i,\varepsilon_P)$ involves an implicit knowledge of the direction of the stress deviator σ'_{ij} due to the particular form of (2.2). It is preferable to modify (2.4) to a form which does not involve σ'_{ij}, and we do this by introducing the potential energy functional $\Psi(u_i,\varepsilon_{ij}^P)$ such that

$$\Psi\left(u_i,\varepsilon_{ij}^P\right)=\int_\Omega U\left(\varepsilon_{ij},\varepsilon_{ij}^P,\varepsilon_{ij,k}^P\right)dx-\int_{S_T}\left(T_i^0 u_i+t_{ij}^0\varepsilon_{ij}^P\right)dS \quad (2.5a)$$

where

$$U\left(\varepsilon_{ij},\varepsilon_{ij}^P,\varepsilon_{ij,k}^P\right)\equiv\frac{1}{2}L_{ijkl}\left(\varepsilon_{ij}-\varepsilon_{ij}^P\right)\left(\varepsilon_{kl}-\varepsilon_{kl}^P\right)+V\left(\varepsilon_{ij}^P,\varepsilon_{ij,k}^P\right). \quad (2.5b)$$

The potential V is taken as

$$V(\varepsilon_{ij}^P,\varepsilon_{ij,k}^P)=\int_0^{E_P}f(e)\,de \quad (2.5c)$$

with the definition (2.3) for E_P replaced by

$$E_P\equiv\sqrt{\tfrac{2}{3}}\left[\varepsilon_{ij}^P\varepsilon_{ij}^P+\lambda^2\varepsilon_{ij,k}^P\varepsilon_{ij,k}^P\right]^{1/2}=\left[\varepsilon_P{}^2+\tfrac{2}{3}\lambda^2\varepsilon_{ij,k}^P\varepsilon_{ij,k}^P\right]^{1/2} \quad (2.6)$$

Thus, the relation (2.2) is abandoned and the plastic strain tensor ε_{ij}^P is treated as an independent variable on an equal footing with the displacement u_i. In the absence of the strain gradient term $(\lambda = 0)$, E_P reduces to the effective plastic strain ε_P, and the stress versus plastic strain relation for uniaxial tension or compression would be $\sigma = f(\varepsilon_P)$. As far as the general relation (2.5b) is concerned, the medium may be compressible but in the sequel it will be taken to be both elastically and plastically incompressible; the modifications to the general formulae about to be given will be mentioned at the time.

Introduce now the conjugate variables

$$\sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}} = L_{ijkl}\left(\varepsilon_{kl} - \varepsilon_{kl}^P\right) \tag{2.7a}$$

$$s_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}^P} = -L_{ijkl}\left(\varepsilon_{kl} - \varepsilon_{kl}^P\right) + \frac{\partial V}{\partial \varepsilon_{ij}^P} \tag{2.7b}$$

$$\tau_{ijk} = \frac{\partial U}{\partial \varepsilon_{ij,k}^P} = \frac{\partial V}{\partial \varepsilon_{ij,k}^P} \tag{2.7c}$$

The principle of virtual work for this medium follows directly from setting to zero the first variation of (2.5a),

$$\int_\Omega \left[\sigma_{ij}\delta\varepsilon_{ij} + s_{ij}\delta\varepsilon_{ij}^P + \tau_{ijk}\delta\varepsilon_{ij,k}^P\right]dx = \int_{S_T}\left[T_i^0\delta u_i + t_{ij}^0\delta\varepsilon_{ij}^P\right]dS \tag{2.8}$$

which, upon integration by parts, provides

$$\int_\Omega \left[-\sigma_{ij,j}\delta u_i + \left(s_{ij} - \tau_{ijk,k}\right)\delta\varepsilon_{ij}^P\right]dx + \int_{S_T}\left[\left(\sigma_{ij}n_j - T_i^0\right)\delta u_i + \left(\tau_{ijk}n_k - t_{ij}^0\right)\delta\varepsilon_{ij}^P\right]\, dS = 0 \tag{2.9}$$

The state of stress and strain in the body of volume Ω and surface S, subjected over a portion S_U of its boundary to prescribed displacements $u_i = u_i^0$ and plastic strains $\varepsilon_{ij}^P = \varepsilon_{ij}^{P0}$, and over the complementary portion S_T to the tractions $\sigma_{ij}n_j = T_i^0$ and 'higher-order tractions' $\tau_{ijk}n_k = t_{ij}^0$, is the unique configuration that minimises the potential energy (2.5a) if (as we assume) L_{ijkl} is positive definite and the potential V is strictly convex. The associated Euler equations that define the solution of the problem are the constitutive relations (2.7), together with the equilibrium relations

$$\sigma_{ij,j} = 0 \tag{2.10a}$$

and

$$\sigma_{ij} + \tau_{ijk,k} = \frac{\partial V}{\partial \varepsilon_{ij}^P} \tag{2.10b}$$

It is noted in passing that the above theory collapses to the conventional deformation theory version of plasticity when there is no dependence on $\varepsilon_{ij,k}$; the higher order stress is then also absent. The plastic strain ε_{ij}^P can then be expressed directly in terms of ε_{ij} via (2.7a) and (2.10b), and the potential function *U*, thus expressed in terms of ε_{ij} alone, becomes a function $w(\varepsilon_{ij})$. The dependence of *U* on $\varepsilon_{ij,k}^P$ through the internal variable E_P, or otherwise, is the essential ingredient of the strain gradient theory.

3. Homogenisation of a non-linear composite

Consider an elastoplastic composite subjected to loading which varies spatially in a sufficiently slow manner with respect to the microstructure that its homogenised response is without strain gradient effects on the macroscale. Thus, its response is described by a potential $U^{eff}\left(\varepsilon_{ij}, \varepsilon_{ij}^P\right)$, which can be determined by considering a special problem, in which the boundary of the composite is subjected to the displacement $u_i = \varepsilon_{ij}^0 x_j$. It is convenient to choose units of length so that the domain Ω has unit volume; then, its total potential is identical with the volume-averaged potential. For this problem, the minimum value of the functional Ψ in (2.5) becomes

$$w^{eff}\left(\varepsilon_{ij}^0\right) = \inf_{\left(u_i, \varepsilon_{ij}^P\right)} \int_\Omega \left[U\left(\varepsilon_{ij}, \varepsilon_{ij}^P, \varepsilon_{ij,k}^P\right)\right] dx \tag{3.1}$$

Here, the infimum is taken over displacement fields u_i that take the prescribed boundary values. The plastic strain fields ε_{ij}^P are unrestricted, corresponding to the natural boundary condition $\tau_{ijk} n_k = 0$ on the whole boundary *S*; this choice is made for convenience, and the particular choice has an influence over a negligible boundary layer of the domain Ω. In order to obtain bounds and estimates for $w^{eff}\left(\varepsilon_{ij}^0\right)$ it is helpful to break down the infimum in (3.1) into two steps:

$$w^{eff}\left(\varepsilon_{ij}^0\right) = \inf_{\varepsilon_{ij}^{P0}} U^{eff}\left(\varepsilon_{ij}^0, \varepsilon_{ij}^{P0}\right) \tag{3.2}$$

where the effective potential U^{eff} is given by

$$U^{eff}\left(\varepsilon_{ij}^0, \varepsilon_{ij}^{P0}\right) = \inf_{\left(u_i, \varepsilon_{ij}^P\right) \in K} \int_\Omega \left[U\left(\varepsilon_{ij}, \varepsilon_{ij}^P, \varepsilon_{ij,k}^P\right)\right] dx \tag{3.3}$$

in which the set *K* is defined as

$$K = \left\{ \left(\boldsymbol{u}, \boldsymbol{\varepsilon}^{P}\right) : u_i = \varepsilon_{ij}^{0} x_j \text{ on } S, \ \left\langle \varepsilon_{ij}^{P} \right\rangle = \varepsilon_{ij}^{P0} \right\} \tag{3.4}$$

Here, the angled bracket implies the mean value over Ω of the quantity concerned.

In the special case that the composite is elastically uniform (so that only its plastic response varies), it is shown in Appendix A that U^{eff} takes the form

$$U^{eff}\left(\varepsilon_{ij}^{0}, \varepsilon_{ij}^{P0}\right) = \frac{1}{2}\left(\varepsilon_{ij}^{0} - \varepsilon_{ij}^{P0}\right) L_{ijk}\left(\varepsilon_{kl}^{0} - \varepsilon_{kl}^{P0}\right) + V^{eff}\left(\varepsilon_{ij}^{P0}\right) \tag{3.5}$$

where

$$V^{eff}\left(\varepsilon_{ij}^{P0}\right) = \inf \int_{\Omega} \left[\tfrac{1}{2}\left(\varepsilon_{ij}^{P} - \varepsilon_{ij}^{P0}\right) L_{ijkl}\left(\varepsilon_{kl}^{P} - \varepsilon_{kl}^{P0}\right) - \tfrac{1}{2}\varepsilon_{ij}^{P}\left(L\Gamma L\right)_{ijkl}\varepsilon_{kl}^{P} + V\left(\varepsilon_{ij}^{P}, \varepsilon_{ij,k}^{P}\right)\right] dx \tag{3.6}$$

The infimum in (3.6) is taken over all fields ε_{ij}^{P} that take the prescribed mean value ε_{ij}^{P0}, and the operator Γ is the same as that introduced by Willis (1977). We shall be considering, later, composites for which the potential V in each phase is a homogeneous function of degree (N+1) in ($\varepsilon_{ij}^{P}, \varepsilon_{ij,k}^{P}$) - see (3.9) below. It may be remarked that the integrand in (3.6) is not correspondingly homogeneous, and hence that $V^{eff}\left(\varepsilon_{ij}^{P0}\right)$ will not be a homogeneous function of degree (N+1).

3.1 The elementary Voigt bound

For a general medium, the Voigt bound for U^{eff} is obtained by substituting into (3.3) the admissible fields ε_{ij}^{0} and ε_{ij}^{P0}, both constant. Thus,

$$U^{eff}\left(\varepsilon_{ij}^{0}, \varepsilon_{ij}^{P0}\right) \leq \int_{\Omega} U\left(\varepsilon_{ij}^{0}, \varepsilon_{ij}^{P0}, 0\right) dx \equiv \left\langle U\left(\varepsilon_{ij}^{0}, \varepsilon_{ij}^{P0}, 0\right)\right\rangle = U_V\left(\varepsilon_{ij}^{0}, \varepsilon_{ij}^{P0}\right). \tag{3.7}$$

In the case of elastic uniformity, the result can be given in the form

$$V^{eff}\left(\varepsilon_{ij}^{P0}\right) \leq \int_{\Omega} V\left(\varepsilon_{ij}^{P0}, 0\right) dx \equiv \left\langle V\left(\varepsilon_{ij}^{P0}, 0\right)\right\rangle = V_V\left(\varepsilon_{ij}^{P0}\right) \tag{3.8}$$

For future reference, this bound is given explicitly in the case of a composite with a strain hardening characteristic given by

$$V_r\left(\varepsilon_{ij}^{P}, \varepsilon_{ij,k}^{P}\right) = \frac{\sigma_r e_0}{N+1}\left(\frac{E_P}{e_0}\right)^{N+1} \tag{3.9}$$

for phases $r = 1, 2, \ldots M$. Phase r has volume fraction p_r. The reference strength σ_r is different for each phase, but they share a common hardening index N. The Voigt bound is

$$V_V\left(\varepsilon_{ij}^{P0}\right) = \frac{\sigma_V e_0}{N+1}\left(\frac{\varepsilon_P^{0}}{e_0}\right)^{N+1} \tag{3.10}$$

where

$$\sigma_V = \sum_{r=1}^{M} p_r \sigma_r \tag{3.11}$$

and ε_P^0 is the von Mises equivalent plastic strain corresponding to ε_{ij}^{P0}, such that $\varepsilon_P^0 = \sqrt{\frac{2}{3}\varepsilon_{ij}^{P0}\varepsilon_{ij}^{P0}}$.

3.2 The complementary energy principle

Designate by $\mathcal{E}$ the minimum value of the energy functional $\Psi\left(u_i, \varepsilon_{ij}^P\right)$, and introduce the complementary energy density

$$U^*\left(\sigma_{ij}, s_{ij}, \tau_{ijk}\right) = \sup_{\varepsilon_{ij}, \varepsilon_{ij}^P, \gamma_{ijk}} \left\{\sigma_{ij}\varepsilon_{ij} + s_{ij}\varepsilon_{ij}^P + \tau_{ijk}\gamma_{ijk} - U\left(\varepsilon_{ij}, \varepsilon_{ij}^P, \gamma_{ijk}\right)\right\}. \tag{3.12}$$

It follows that

$$\begin{aligned} \mathcal{E} &= \inf\left\{ \int_\Omega U\left(\varepsilon_{ij}, \varepsilon_{ij}^P, \varepsilon_{ij,k}^P\right)dx - \int_{S_T}\left(T_i^0 u_i + t_{ij}^0 \varepsilon_{ij}^P\right) dS \right\} \\ &\geq \inf\left\{ \int_\Omega \left[\sigma_{ij}\varepsilon_{ij} + s_{ij}\varepsilon_{ij}^P + \tau_{ijk}\varepsilon_{ij,k}^P - U^*\left(\sigma_{ij}, s_{ij}, \tau_{ijk}\right)\right]dx - \int_{S_T}\left(T_i^0 u_i + t_{ij}^0 \varepsilon_{ij}^P\right) dS \right\} \\ &= \inf\left\{ \int_\Omega \left[-\sigma_{ij,j}u_i + \left(s_{ij} - \tau_{ijk,k}\right)\varepsilon_{ij}^P - U^*\left(\sigma_{ij}, s_{ij}, \tau_{ijk}\right)\right] dx \right. \\ &\qquad \left. + \int_S \left[\sigma_{ij} n_j u_i + \tau_{ijk} n_k \varepsilon_{ij}^P\right]dS - \int_{S_T}\left(T_i^0 u_i + t_{ij}^0 \varepsilon_{ij}^P\right) dS \right\} \end{aligned} \tag{3.13}$$

The infimum in each line above is taken over the admissible fields u_i, ε_{ij}^P that satisfy the conditions $u_i = u_i^0$ and $\varepsilon_{ij}^P = \varepsilon_{ij}^{P0}$ on S_U. The inequality is true for all σ_{ij}, s_{ij} and τ_{ijk}, and hence it is optimised by appropriate choice of these fields. Avoidance of the value $-\infty$ requires that these fields should satisfy the equations

$$\sigma_{ij,j} = 0 \quad \text{and} \quad \tau_{ijk,k} - s_{ij} = 0 \quad \text{in } \Omega \tag{3.14}$$

together with the boundary conditions

$$\sigma_{ij} n_j = T_i^0 \ \text{and}\ \tau_{ijk} n_k = t_{ij}^0 \ \text{ over } S_T \ . \tag{3.15}$$

The inequality then gives

$$\varepsilon \geq \sup_{\sigma_{ij}, s_{ij}, \tau_{ijk}} \left\{ \int_{S_U} \left[\sigma_{ij} n_j u_i^0 + \tau_{ijk} n_k \varepsilon_{ij}^{P0}\right]dS - \int_\Omega \left[U^*\left(\sigma_{ij}, s_{ij}, \tau_{ijk}\right)\right] dx\right\} \tag{3.16}$$

Here, the supremum is taken over fields that satisfy the restrictions (3.14) and (3.15). It is easy to verify that the supremum is realised by the actual solution, and the value of the supremum is $\mathcal{E}$, so long as the potential U is convex and suitably smooth. The relation (3.16) is an expression of the complementary energy principle for this system.

3.3 Complementary characterisation of the effective response and the Reuss bound

The method just employed to develop the complementary principle, dual to the minimum energy principle, can also be employed to obtain a dual characterisation of the effective response of a composite. Starting from the "primal" definition (3.3) and following the reasoning given in equations (3.13) leads to the inequality

$$U^{eff}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right) \geq \inf_{\left(u_i,\varepsilon_{ij}^{P}\right)\in K} \left\{ \int_{\Omega}\left[-\sigma_{ij,j}u_i + \left(s_{ij}-\tau_{ijk,k}\right)\varepsilon_{ij}^{P} - U^{*}\left(\sigma_{ij},s_{ij},\tau_{ijk}\right)\right]dx \right.$$

$$\left. + \int_{S}\left(\sigma_{ij}n_j\left(\varepsilon_{ik}^{0}x_k\right)+\tau_{ijk}n_k\varepsilon_{ij}^{P}\right)dS \right\} . \qquad (3.17)$$

where the set K has already been specified by (3.4). The value $-\infty$ is avoided if

$$\sigma_{ij,j}=0 \text{ and } \tau_{ijk,k}=s_{ij}-s_{ij}^{0} \text{ in } \Omega, \text{ and } \tau_{ijk}n_k=0 \text{ over } S, \qquad (3.18)$$

where s_{ij}^{0} is any constant, equal, in fact, to the mean value of s_{ij} over Ω. When these conditions are met, the inequality (3.17) reduces to

$$U^{eff}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right) \geq \sigma_{ij}^{0}\varepsilon_{ij}^{0} + s_{ij}^{0}\varepsilon_{ij}^{P0} - \int_{\Omega}U^{*}\left(\sigma_{ij},s_{ij},\tau_{ijk}\right)dx. \qquad (3.19)$$

It follows that

$$U^{eff}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right) \geq \sup_{\sigma_{ij}^{0},\, s_{ij}^{0}} \left\{\sigma_{ij}^{0}\varepsilon_{ij}^{0} + s_{ij}^{0}\varepsilon_{ij}^{P0} - U^{*eff}\left(\sigma_{ij}^{0},s_{ij}^{0}\right)\right\}, \qquad (3.20)$$

where

$$U^{*eff}\left(\sigma_{ij}^{0},s_{ij}^{0}\right) = \inf\int_{\Omega}U^{*}\left(\sigma_{ij},s_{ij},\tau_{ijk}\right)dx, \qquad (3.21)$$

the infimum being taken over fields $\sigma_{ij}, s_{ij}, \tau_{ijk}$ that satisfy equations (3.18) and σ_{ij} has mean value σ_{ij}^{0} .

An elementary bound, of Reuss type, for U^{*eff} follows by substituting into (3.21) the admissible fields $\sigma_{ij}=\sigma_{ij}^{0}, s_{ij}=s_{ij}^{0}, \tau_{ijk}=0$. This gives

$$U^{*eff}\left(\sigma_{ij}^{0},s_{ij}^{0}\right) \leq \int_{\Omega}U^{*}\left(\sigma_{ij}^{0},s_{ij}^{0},0\right)dV \equiv \left\langle U^{*}\left(\sigma_{ij}^{0},s_{ij}^{0},0\right)\right\rangle = U_{R}^{*}\left(\sigma_{ij}^{0},s_{ij}^{0}\right) \qquad (3.22)$$

and hence via (3.20),

$$U^{eff}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right) \geq \left(U_{R}^{*}\right)^{*}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right). \qquad (3.23)$$

The potentials $V\left(\varepsilon_{ij}^{P},\gamma_{ijk}\right)$ considered in this work satisfy the reasonable condition $V\left(\varepsilon_{ij}^{P},0\right)\leq V\left(\varepsilon_{ij}^{P},\gamma_{ijk}\right)$ for all γ_{ijk}. Assuming in addition that the elastic moduli L_{ijkl} are constant, elementary calculation reduces the Reuss bound (3.23) to the simpler form

$$V^{eff}\left(\varepsilon_{ij}^{P0}\right) \geq V_R\left(\varepsilon_{ij}^{P0}\right) \tag{3.24}$$

where $V_R^*\left(s_{ij}^0\right)=\left\langle V^*\left(s_{ij}^0,0\right)\right\rangle$, and $V_R\equiv\left(V_R^*\right)^*$ is the dual of V_R^*. When the potential V takes the form (3.9) in phase r, this bound becomes

$$V_R^*\left(s_{ij}^0\right)=\frac{N}{N+1}\sigma_R e_0\left(\frac{s_e^0}{\sigma_R}\right)^{(N+1)/N}, \quad V_R\left(\varepsilon_{ij}^{P0}\right)=\frac{\sigma_R e_0}{N+1}\left(\frac{\varepsilon_P^0}{e_0}\right)^{N+1}, \tag{3.25}$$

where s_e^0 is the equivalent stress associated with s_{ij}^0 and

$$\sigma_R=\left(\sum_{r=1}^{M}p_r\,\sigma_r^{-1/N}\right)^{-N}. \tag{3.26}$$

In the elastic-ideally plastic limit $N\rightarrow 0$ and (3.26) simplifies to $\sigma_R=\min\{\sigma_r\}$ over all phases r =1,2,...M of the composite.

4. Linear Hashin-Shtrikman bounds and self-consistent estimate

4.1 The upper bound

Bounds (and estimates) for the energy of a linear composite can be obtained by introducing a linear comparison solid. We shall restrict our attention to the incompressible isotropic case, such that $\varepsilon_{ii}=\varepsilon_{ii}^{P}=0$. For a linear strain gradient solid define the energy density U as

$$U\left(\varepsilon_{ij},\varepsilon_{ij}^{P},\varepsilon_{ij,k}^{P}\right)=\mu\left(\varepsilon_{ij}-\varepsilon_{ij}^{P}\right)\left(\varepsilon_{ij}-\varepsilon_{ij}^{P}\right)+\frac{1}{2}bE_P{}^2 \tag{4.1}$$

in terms of a spatially uniform shear modulus μ, a spatially non-uniform stiffness $b(x)$, and the choice (2.6) for E_P. Now introduce a linear comparison solid with potential

$$U_0\left(\varepsilon_{ij},\varepsilon_{ij}^{P},\varepsilon_{ij,k}^{P}\right)=\mu\left(\varepsilon_{ij}-\varepsilon_{ij}^{P}\right)\left(\varepsilon_{ij}-\varepsilon_{ij}^{P}\right)+\frac{1}{2}b_0E_P{}^2 \tag{4.2}$$

and note that $U-U_0$ is a function of only $\left(\varepsilon_{ij}^{P},\varepsilon_{ij,k}^{P}\right)$.

We begin by considering the Hashin-Shtrikman upper bound. Choose U_0 in such a way that at each point of the composite $\left(U_0-U\right)$ grows faster than linearly when ε_{ij}^{P} or $\varepsilon_{ij,k}^{P}$ is large.

Later, this shall be made explicit by choosing $b_0(\boldsymbol{x}) \geq b(\boldsymbol{x})$ throughout the composite. Define the dual $(U-U_0)_*(\alpha_{ij}, \beta_{ijk})$ as

$$(U-U_0)_*(\alpha_{ij}, \beta_{ijk}) = \inf_{\varepsilon_{ij}^P, \gamma_{ijk}} \left[\alpha_{ij}\varepsilon_{ij}^P + \beta_{ijk}\,\gamma_{ijk} - (U-U_0)\left(\varepsilon_{ij}^P, \gamma_{ijk}\right)\right] \tag{4.3}$$

where we take $\alpha_{ii} = 0$ and $\beta_{iik} = 0$ to match the incompressibility requirement. The infimum is attained at

$$\alpha_{ij} = \frac{\partial}{\partial \varepsilon_{ij}^P}(U-U_0) = \frac{2}{3}(b-b_0)\varepsilon_{ij}^P \tag{4.4a}$$

and

$$\beta_{ijk} = \frac{\partial}{\partial \gamma_{ijk}}(U-U_0) = \frac{2}{3}\lambda^2(b-b_0)\gamma_{ijk} \tag{4.4b}$$

and a direct evaluation of $(U-U_0)_*$ gives

$$(U-U_0)_*(\alpha_{ij}, \beta_{ijk}) = \frac{3}{4}\frac{\alpha_{ij}\alpha_{ij}}{b-b_0} + \frac{3}{4}\frac{\beta_{ijk}\beta_{ijk}}{(b-b_0)\lambda^2} \tag{4.5}$$

Fenchel's inequality can be stated as

$$U\left(\varepsilon_{ij}, \varepsilon_{ij}^P, \varepsilon_{ij,k}^P\right) \leq \alpha_{ij}\varepsilon_{ij}^P + \beta_{ijk}\,\varepsilon_{ij,k}^P + U_0 - (U-U_0)_*(\alpha_{ij}, \beta_{ijk}) \tag{4.6}$$

for all values of $\left(\varepsilon_{ij}, \varepsilon_{ij}^P, \varepsilon_{ij,k}^P\right)$ and $(\alpha_{ij}, \beta_{ijk})$, with equality attained when $(\alpha_{ij}, \beta_{ijk})$ satisfy (4.4).

We shall be concerned with the effective properties of a composite subjected to a uniform macroscopic strain ε_{ij}^0 and macroscopic average plastic strain ε_{ij}^{P0}, and in the absence of body forces. To simplify the bounds and estimates, we shall limit our solution space by making the choice $\beta_{ijk} \equiv 0$ *pointwise*[1]. For current purposes, it is sufficient to take the linear comparison medium as uniform spatially, and we shall do so in the sequel.

[1] We expect that a full calculation would in any case give $\beta_{ijk} = 0$.

The effective energy U^{eff} of the composite, subjected to the macroscopic strain state $\left(\varepsilon_{ij}^{0}, \varepsilon_{ij}^{P0}\right)$ is given by

$$U^{eff}\left(\varepsilon_{ij}^{0}, \varepsilon_{ij}^{P0}\right) = \inf_{\left(u_i, \varepsilon_{ij}^{P}\right) \in K} \int_{\Omega} \left[U\left(\varepsilon_{ij}, \varepsilon_{ij}^{P}, \varepsilon_{ij,k}^{P}\right)\right] dx \tag{4.7}$$

in which the set K is $\left\{\left(\boldsymbol{u}, \boldsymbol{\varepsilon}^{P}\right): u_i = \varepsilon_{ij}^{0} x_j, \left\langle \varepsilon_{ij}^{P} \right\rangle = \varepsilon_{ij}^{P0}\right\}$, as has already been defined in (3.4).

In order to construct an upper bound for U^{eff} we substitute (4.5) and (4.6) into (4.7), and write, for any choice of α_{ij} and $\beta_{ijk} \equiv 0$,

$$U^{eff}\left(\varepsilon_{ij}^{0}, \varepsilon_{ij}^{P0}\right) \leq \inf_{\left(u_i, \varepsilon_{ij}^{P}\right) \in K} \int_{\Omega} \left[U_0\left(\varepsilon_{ij}, \varepsilon_{ij}^{P}, \varepsilon_{ij,k}^{P}\right) + \alpha_{ij}\varepsilon_{ij}^{P} - \frac{3}{4}\frac{\alpha_{ij}\alpha_{ij}}{b - b_0}\right] dx \tag{4.8}$$

The infimum of (4.8) is obtained for any choice of $\alpha_{ij}(\boldsymbol{x})$ by taking variations of the right hand side (RHS) of (4.8) with respect to $\left(u_i, \varepsilon_{ij}^{P}\right) \in K$, to obtain the Principle of Virtual Work

$$\int_{\Omega} \left\{ \left[2\mu\left(\varepsilon_{ij} - \varepsilon_{ij}^{P}\right)\right] \delta\varepsilon_{ij} + \left[-2\mu\left(\varepsilon_{ij} - \varepsilon_{ij}^{P}\right) + \alpha_{ij} + \frac{2}{3} b_0 \varepsilon_{ij}^{P}\right] \delta\varepsilon_{ij}^{P} + \left[\frac{2}{3} b_0 \lambda^2 \varepsilon_{ij,k}^{P}\right] \delta\varepsilon_{ij,k}^{P} \right\} dx = 0 \tag{4.9}$$

and thereby the equilibrium equations,

$$\left(2\mu + \frac{2}{3} b_0\right)\left(\varepsilon_{ij}^{P} - \varepsilon_{ij}^{P0}\right) - \frac{2}{3} b_0 \lambda^2 \varepsilon_{ij,kk}^{P} + \left(\alpha_{ij} - \left\langle \alpha_{ij} \right\rangle\right) - 2\mu\left(\varepsilon_{ij} - \varepsilon_{ij}^{0}\right) = 0 \tag{4.10a}$$

and

$$\mu u_{i,jj} - 2\mu \varepsilon_{ij,j}^{P} - p_{,i} = 0 \tag{4.10b}$$

where $p(\boldsymbol{x})$ is an equilibrating pressure field.

Equations (4.10) are linear, and so their solution can be constructed as follows. First, solve the equations with the polarisation α_{ij} absent in (4.10); the solution is $\left(u_i^{0}, \varepsilon_{ij}^{0}, \varepsilon_{ij}^{P0}\right)$, with $u_i^{0} = \varepsilon_{ij}^{0} x_j$. It satisfies the virtual work relation,

$$\int_{\Omega} \left\{ \left[2\mu\left(\varepsilon_{ij}^{0} - \varepsilon_{ij}^{P0}\right)\right] \delta\varepsilon_{ij} + \left[-2\mu\left(\varepsilon_{ij}^{0} - \varepsilon_{ij}^{P0}\right) + \frac{2}{3} b_0 \varepsilon_{ij}^{P0}\right] \delta\varepsilon_{ij}^{P} \right\} dx = 0 \tag{4.11}$$

Second, solve the equations with the polarisation α_{ij} present in (4.10), but with zero mean fields applied, and call this solution $\left(\tilde{u}_i, \tilde{\varepsilon}_{ij}, \tilde{\varepsilon}_{ij}^{P}\right)$, defined over all of Ω. This solution satisfies the virtual work relation,

$$\int_{\Omega}\left\{\left[2\mu\left(\tilde{\varepsilon}_{ij}-\tilde{\varepsilon}_{ij}^{P}\right)\right]\delta\varepsilon_{ij}+\left[-2\mu\left(\tilde{\varepsilon}_{ij}-\tilde{\varepsilon}_{ij}^{P}\right)+\alpha_{ij}+\frac{2}{3}b_0\tilde{\varepsilon}_{ij}^{P}\right]\delta\varepsilon_{ij}^{P}+\left[\frac{2}{3}b_0\lambda^2\tilde{\varepsilon}_{ij,k}^{P}\right]\delta\varepsilon_{ij,k}^{P}\right\}dx=0 \quad (4.12)$$

where $\left(\tilde{u}_i,\tilde{\varepsilon}_{ij},\tilde{\varepsilon}_{ij}^{P}\right)$ are derived from the set of fields, $\left\{\left\langle\tilde{\varepsilon}_{ij}\right\rangle=0,\left\langle\tilde{\varepsilon}_{ij}^{P}\right\rangle=0\right\}$. Then, the solution to (4.10) is given by

$$u_i=u_i^0+\tilde{u}_i\ , \qquad \varepsilon_{ij}=\varepsilon_{ij}^0+\tilde{\varepsilon}_{ij} \qquad \text{and} \qquad \varepsilon_{ij}^{P}=\varepsilon_{ij}^{P0}+\tilde{\varepsilon}_{ij}^{P} \qquad (4.13)$$

The infimum relation (4.8) can be simplified by making use of (4.11) and (4.12), as follows. First, re-write (4.11) and (4.12) by making the choice $\delta\varepsilon_{ij}=\tilde{\varepsilon}_{ij}$ and $\delta\varepsilon_{ij}^{P}=\tilde{\varepsilon}_{ij}^{P}$, and second, expand the integral expression for $U_0\left(\varepsilon_{ij},\varepsilon_{ij}^{P},\varepsilon_{ij,k}^{P}\right)+\alpha_{ij}\varepsilon_{ij}^{P}$ by making use of (4.2) and (4.13), to obtain

$$\int_{\Omega}\left\{U_0\left(\varepsilon_{ij},\varepsilon_{ij}^{P},\varepsilon_{ij,k}^{P}\right)+\alpha_{ij}\varepsilon_{ij}^{P}\right\}dx=\int_{\Omega}\left\{U_0\left(\varepsilon_{ij}^0,\varepsilon_{ij}^{P0},0\right)+\frac{1}{2}\alpha_{ij}\left(\varepsilon_{ij}^{P}-\varepsilon_{ij}^{P0}\right)+\alpha_{ij}\varepsilon_{ij}^{P0}\right\}dx \qquad (4.14)$$

Consequently, (4.8) can be expressed in the form

$$U^{eff}\left(\varepsilon_{ij}^0,\varepsilon_{ij}^{P0}\right)\leq U_0\left(\varepsilon_{ij}^0,\varepsilon_{ij}^{P0},0\right)+\int_{\Omega}\left[\ \alpha_{ij}\varepsilon_{ij}^{P0}-\frac{3}{4}\frac{\alpha_{ij}\alpha_{ij}}{b-b_0}+\frac{1}{2}\alpha_{ij}\left(\varepsilon_{ij}^{P}-\varepsilon_{ij}^{P0}\right)\right]dx \qquad (4.15)$$

where ε_{ij}^{P} is obtained by solving the system (4.10), subject to the boundary condition $u_i=\varepsilon_{ij}^0x_j$ on *S* and to the constraint $\left\langle\varepsilon_{ij}^{P}\right\rangle=\varepsilon_{ij}^{P0}$. We proceed by solving the governing partial differential equations (4.10a,b) in order to obtain the following integral expression for $\left(\varepsilon_{ij}^{P}-\varepsilon_{ij}^{P0}\right)$ in terms of $\left(\alpha_{ij}-\left\langle\alpha_{ij}\right\rangle\right)$:

$$\left(\varepsilon_{ij}^{P}-\varepsilon_{ij}^{P0}\right)(\mathbf{x})=-\int\left[\Lambda_{ijkl}(\mathbf{x}-\mathbf{x}')\left(\alpha_{kl}-\left\langle\alpha_{kl}\right\rangle\right)(\mathbf{x}')\right]dx' \qquad (4.16)$$

The integration over the domain Ω is approximated by an integral over all space because $\left(\alpha_{ij}-\left\langle\alpha_{ij}\right\rangle\right)$ oscillates rapidly and has mean value zero. The derivation of the integral operator $\Lambda_{ijkl}(\mathbf{x}-\mathbf{x}')$ by Fourier Transforms is given in Appendix B: the Fourier Transform $\hat{\Lambda}_{ijkl}(\boldsymbol{\xi})$ is stated explicitly by relation (B15). Optimisation of (4.15) with respect to $\boldsymbol{\alpha_{ij}}$, and noting the linear relation (4.16) between $\left(\varepsilon_{ij}^{P}-\varepsilon_{ij}^{P0}\right)$ and $\left(\alpha_{ij}-\left\langle\alpha_{ij}\right\rangle\right)$, gives at the solution for $\boldsymbol{\alpha_{ij}}$,

$$U^{eff}\left(\varepsilon_{ij}^0,\varepsilon_{ij}^{P0}\right)=U_0\left(\varepsilon_{ij}^0,\varepsilon_{ij}^{P0},0\right)+\frac{1}{2}\left\langle\alpha_{ij}\right\rangle\varepsilon_{ij}^{P0} \qquad (4.17)$$

4.2 The statistics of the composite and the Hashin-Shtrikman approximation for the polarisation field

Consider an *M*-phase composite, with modulus $b^{(r)}$ in each phase $r = 1,2,\ldots.M$. Then, $b(\boldsymbol{x})$ takes the value $b^{(r)}$ if $\boldsymbol{x}$ lies in material of type *r*, and we may write

$$b(\boldsymbol{x}) = \sum_{r=1}^{M} b^{(r)} \chi^{(r)}(\boldsymbol{x}) \tag{4.18}$$

where $\chi^{(r)}(\boldsymbol{x})$ is the characteristic function of the region occupied by material *r*, taking the value 1 if material *r* is at $\boldsymbol{x}$ and zero otherwise. In our search for the Hashin-Shtrikman upper bound we shall assume that the modulus b_0 in the linear comparison solid equals the largest of the values for $b^{(r)}$.

Now admit that the composite is a random medium so that $\chi^{(r)}(\boldsymbol{x})$ become random fields. Then, the probability $p_r(\boldsymbol{x})$ that material *r* is at $\boldsymbol{x}$ is given by the ensemble average of $\chi^{(r)}(\boldsymbol{x})$, written as

$$p_r(\boldsymbol{x}) = \left\langle \chi^{(r)}(\boldsymbol{x}) \right\rangle \tag{4.19}$$

Similarly, the probability of finding simultaneously material of type *r* at $\boldsymbol{x}$ and type *s* at $\boldsymbol{x}'$ is

$$p_{rs}(\boldsymbol{x}, \boldsymbol{x}') = \left\langle \chi^{(r)}(\boldsymbol{x}) \chi^{(s)}(\boldsymbol{x}') \right\rangle \tag{4.20}$$

We shall take the composite to be statistically uniform (i.e. the statistics are those of a stationary random process), such that $p_r(\boldsymbol{x})$ is independent of the position $\boldsymbol{x}$, and equals the volume fraction p_r of phase r; likewise, $p_{rs}(\boldsymbol{x}, \boldsymbol{x}')$ depends only upon the relative position $(\boldsymbol{x} - \boldsymbol{x}')$ and can be rewritten as $p_{rs}(\boldsymbol{x} - \boldsymbol{x}')$. In this initial study we shall consider a *statistically isotropic composite*, such that the $p_{rs}(\boldsymbol{x} - \boldsymbol{x}')$ depends only upon the radial separation $|\boldsymbol{x} - \boldsymbol{x}'|$. With the further assumption that each phase is isotropic, the overall effective properties of the composite are guaranteed to be isotropic.

Now make the Hashin-Shtrikman type approximation that the polarisation α_{ij} has the uniform value $\alpha_{ij}^{(r)}$ within each phase r, so that we may write

$$\alpha_{ij}(\boldsymbol{x}) = \sum_{r=1}^{M} \alpha_{ij}^{(r)} \chi^{(r)}(\boldsymbol{x}) \tag{4.21}$$

With this approximation in place, the inequality (4.15) can be ensemble averaged to read[2]

$$U^{eff}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right)\le\left[\begin{array}{l}U_0\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0},0\right)+\sum_{r=1}^{M}\alpha_{ij}^{(r)}p_r\varepsilon_{ij}^{P0}-\frac{3}{4}\sum_{r=1}^{M}p_r\frac{\alpha_{ij}^{(r)}\alpha_{ij}^{(r)}}{b^{(r)}-b_0}\\-\frac{1}{2}\sum_{r=1}^{M}\sum_{s=1}^{M}\alpha_{ij}^{(r)}\alpha_{ij}^{(s)}\int\left[\Lambda_{ijkl}(\boldsymbol{x})\,\{p_{rs}(\boldsymbol{x})-p_rp_s\}\right]dx\,.\end{array}\right] \tag{4.22}$$

Now choose the best set of values of $\alpha_{ij}^{(r)}$: taking the variation of (4.22) with respect to $\alpha_{ij}^{(r)}$ leads to

$$p_r\varepsilon_{ij}^{P0}=\frac{3}{2}\frac{p_r\alpha_{ij}^{(r)}}{b^{(r)}-b_0}+\sum_{s=1}^{M}\alpha_{kl}^{(s)}\int\left[\Lambda_{ijkl}(\boldsymbol{x})\{p_{rs}(\boldsymbol{x})-p_rp_s\}\right]dx \tag{4.23}$$

for r =1,2,...M.

Additional assumption: the two phase composite

At this stage in the calculation we specialise to the case of a two phase medium, M =2, with isotropic statistics in order to simplify the second term on the right hand side of (4.23).

First, note the connections

$$p_{11}(\boldsymbol{x})-p_1p_1=-(p_{12}(\boldsymbol{x})-p_1p_2)=-(p_{21}(\boldsymbol{x})-p_1p_2)=p_{22}(\boldsymbol{x})-p_2p_2=p_1p_2h(r) \tag{4.24}$$

where the correlation functions can be expressed by a specified radial function $h(r)$ since the statistics are taken to be isotropic. For simplicity, we shall follow the choice made by Smyshlyaev and Fleck (1995) and Drugan (2003), and take

$$h(r)=e^{-r/a} \tag{4.25}$$

so that a is a correlation length scale for the microstructure. (Drugan showed that the exponential two-point correlation function gives excellent agreement with the result calculated from the Verlet-Weis improvement to the Percus-Yevick model, see Figure 1 of Drugan, 2003.)

Second, introduce the fourth order isotropic tensor Q_{ijkl} by

$$p_1p_2\,Q_{ijkl}\equiv\int\left[\Lambda_{ijkl}(\boldsymbol{x})\{p_{22}(\boldsymbol{x})-p_2p_2\}\right]dx=p_1p_2\int\left[\Lambda_{ijkl}(\boldsymbol{x})h(r)\right]\,dx \tag{4.26}$$

Substitution of the identities (4.24) and the definition (4.26) into (4.23) gives

[2] In the 'homogenisation limit' of fine scale microstructure, U^{eff} is the same in every realisation of the composite.

$$\frac{3}{2}\frac{\alpha_{ij}^{(r)}}{b^{(r)}-b_0}+Q_{ijkl}\left(\alpha_{kl}^{(r)}-\langle\alpha_{kl}\rangle\right)=\varepsilon_{ij}^{P0} \qquad , \; r=1,2 \tag{4.27}$$

Recall that the expression (4.17) for the effective energy U^{eff} of the composite is expressed in terms of the average polarisation $\langle\alpha_{ij}\rangle$. The optimal value of the bound (4.22) reduces similarly and so it suffices to solve (4.27) for $\langle\alpha_{ij}\rangle$ rather than for the value $\alpha_{ij}^{(r)}$ in each phase. Routine manipulation provides the result

$$\langle\alpha_{ij}\rangle=\left\{\left\langle\left(\boldsymbol{I}'+\frac{2}{3}\left(b^{(r)}-b_0\right)\boldsymbol{Q}\right)^{-1}\right\rangle\right\}_{ijpq}^{-1}\left\langle\frac{2}{3}\left(b^{(r)}-b_0\right)\left(\boldsymbol{I}'+\frac{2}{3}\left(b^{(r)}-b_0\right)\boldsymbol{Q}\right)_{pqkl}^{-1}\right\rangle\varepsilon_{kl}^{P0} \tag{4.28}$$

where $\boldsymbol{I}'$ is the incompressible identity tensor. Hence, the bound (4.22) implies that

$$U^{eff}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right)\le\mu\left(\varepsilon_{ij}^{0}-\varepsilon_{ij}^{P0}\right)\left(\varepsilon_{ij}^{0}-\varepsilon_{ij}^{P0}\right)+\frac{1}{3}B_{ijkl}^{HS+}\,\varepsilon_{ij}^{P0}\varepsilon_{kl}^{P0} \tag{4.29}$$

where

$$B_{ijkl}^{HS+}\equiv b_0 I'_{ijkl}+\frac{3}{2}\left\{\left\langle\left(\boldsymbol{I}'+\frac{2}{3}\left(b^{(r)}-b_0\right)\boldsymbol{Q}\right)^{-1}\right\rangle\right\}_{ijpq}^{-1}\left\langle\frac{2}{3}\left(b^{(r)}-b_0\right)\left(\boldsymbol{I}'+\frac{2}{3}\left(b^{(r)}-b_0\right)\boldsymbol{Q}\right)_{pqkl}^{-1}\right\rangle \tag{4.30}$$

Since Q_{ijkl} is isotropic it can be expressed in Hill's notation (Hill, 1965) as

$$\left(Q_{ijkl}\right)=\left(3\kappa_Q,2\mu_Q\right) \tag{4.31}$$

where

$$9\kappa_Q=Q_{iijj} \qquad \text{and} \qquad 3\kappa_Q+10\mu_Q=Q_{ijij} \tag{4.32}$$

Note that Q_{ijkl} is a compliance measure, and since the linear comparison solid is incompressible it follows that κ_Q=0. Since Q_{ijkl} and I'_{ijkl} are isotropic, the tensor B_{ijkl}^{HS+} is isotropic, and can be expressed in terms of a single scalar quantity b^{HS+} as

$$B_{ijkl}^{HS+}=b^{HS+}I'_{ijkl} \tag{4.33a}$$

and in Hill's representation as

$$\left(B_{ijkl}^{HS+}\right)=\left(0,b^{HS+}\right) \tag{4.33b}$$

It is now possible to convert (4.30) into the scalar algebraic statement,

$$b^{HS+} = \frac{\sum_{r=1}^{2}\left(\frac{p_r\, b^{(r)}}{1+\frac{4}{3}\mu_Q\left(b^{(r)}-b_0\right)}\right)}{\sum_{r=1}^{2}\left(\frac{p_r}{1+\frac{4}{3}\mu_Q\left(b^{(r)}-b_0\right)}\right)} \quad . \tag{4.34}$$

The upper bound expression (4.29) for the effective energy can be written

$$U^{eff}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right) \le U^{HS+}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right) = \mu\left(\varepsilon_{ij}^{0}-\varepsilon_{ij}^{P0}\right)\left(\varepsilon_{ij}^{0}-\varepsilon_{ij}^{P0}\right)+\frac{1}{3}b^{HS+}\varepsilon_{ij}^{P0}\varepsilon_{ij}^{P0} \tag{4.35}$$

The exact expression (4.17) for U^{eff} can be written similarly

$$U^{eff}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right) = \mu\left(\varepsilon_{ij}^{0}-\varepsilon_{ij}^{P0}\right)\left(\varepsilon_{ij}^{0}-\varepsilon_{ij}^{P0}\right)+\frac{1}{3}b^{eff}\varepsilon_{ij}^{P0}\varepsilon_{ij}^{P0} \quad , \tag{4.36}$$

so that (4.35) gives $b^{eff} \le b^{HS+}$. It remains to obtain an algebraic expression for μ_Q from the definitions (4.32) and (4.26). The derivation is somewhat lengthy, and is given in Appendix C, with a general expression for μ_Q for an arbitrary choice of $h(r)$. Here, we state only the final result upon making the choice (4.25) for $h(r)$,

$$\mu_Q = \frac{3}{10b_0\left(1+\frac{\lambda}{a}\right)^2} + \frac{9}{20\left(3\mu+b_0\right)\left(1+\frac{\lambda}{a}\left(\frac{b_0}{3\mu+b_0}\right)^{1/2}\right)^2} \tag{4.37}$$

4.3 Hashin-Shtrikman lower bound

A lower bound for $U^{eff}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right)$ is obtained by choosing a comparison medium such that $b_0 \le b$ throughout the composite. Then, $\left(U-U_0\right)$ grows faster than linearly at large ε_{ij}^{P} or $\varepsilon_{ij,k}^{P}$, and the following dual function $\left(U-U_0\right)^{*}\left(\alpha_{ij},\beta_{ijk}\right)$ can be defined,

$$\left(U-U_0\right)^{*}\left(\alpha_{ij},\beta_{ijk}\right) = \sup_{\varepsilon_{ij}^{P},\gamma_{ijk}} \left[\alpha_{ij}\varepsilon_{ij}^{P} + \beta_{ijk}\gamma_{ijk} - \left(U-U_0\right)\left(\varepsilon_{ij}^{P},\gamma_{ijk}\right)\right] \tag{4.38}$$

Again, we take $\alpha_{ii}=0$ and $\beta_{iik}=0$ to match the incompressibility requirement. Note the close connection between $\left(U-U_0\right)^{*}\left(\alpha_{ij},\beta_{ijk}\right)$ and $\left(U-U_0\right)_{*}\left(\alpha_{ij},\beta_{ijk}\right)$, as introduced by (4.3).

Fenchel's inequality provides

$$U\left(\varepsilon_{ij},\varepsilon_{ij}^{P},\varepsilon_{ij,k}^{P}\right) \geq \alpha_{ij}\varepsilon_{ij}^{P} + \beta_{ijk}\,\varepsilon_{ij,k}^{P} + U_0 - \left(U - U_0\right)^{*}\left(\alpha_{ij},\beta_{ijk}\right) \tag{4.39}$$

for all values of $\left(\varepsilon_{ij},\varepsilon_{ij}^{P},\varepsilon_{ij,k}^{P}\right)$ and $\left(\alpha_{ij},\beta_{ijk}\right)$. The lower bound effective energy U^{HS-} of the composite, subjected to the macroscopic strain state $\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right)$ is derived by the same argument as that used to obtain (4.8) from (4.4), giving

$$U^{eff}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right) \geq \inf_{\left(u_i,\varepsilon_{ij}^{P}\right)\in K} \int_{\Omega}\left[U_0\left(\varepsilon_{ij},\varepsilon_{ij}^{P},\varepsilon_{ij,k}^{P}\right) + \alpha_{ij}\varepsilon_{ij}^{P} - \frac{3}{4}\frac{\alpha_{ij}\alpha_{ij}}{b-b_0}\right]dx \tag{4.40}$$

where the set K is again $\left\{\left(\boldsymbol{u},\boldsymbol{\varepsilon}^{P}\right): u_i = \varepsilon_{ij}^{0}x_j, \left\langle\varepsilon_{ij}^{P}\right\rangle = \varepsilon_{ij}^{P0}\right\}$. Applying again the above optimisation scheme to the right hand side of (4.40), a lower bound U^{HS-} is obtained, of similar algebraic structure to (4.35), giving

$$U^{HS-}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right) = \mu\left(\varepsilon_{ij}^{0}-\varepsilon_{ij}^{P0}\right)\left(\varepsilon_{ij}^{0}-\varepsilon_{ij}^{P0}\right) + \frac{1}{3}b^{HS-}\varepsilon_{ij}^{P0}\varepsilon_{ij}^{P0} \tag{4.41}$$

where b^{HS-} is given by (4.34), but with b_0 equal to the lower value of modulus $b^{(r)}$ of the two phases.

4.4 Self-consistent estimate

We follow the self-consistent approach of Willis (1977, 1981) by choosing an optimal linear comparison medium. This entails replacing b_0 by b^{eff} in (4.34), so that it becomes an implicit equation for b^{eff}, to be solved by numerical iteration,

$$b^{SC} = \frac{\displaystyle\sum_{r=1}^{2}\left(\frac{p_r\,b^{(r)}}{1+\frac{4}{3}\mu_Q\left(b^{(r)}-b^{SC}\right)}\right)}{\displaystyle\sum_{r=1}^{2}\left(\frac{p_r}{1+\frac{4}{3}\mu_Q\left(b^{(r)}-b^{SC}\right)}\right)} \tag{4.42}$$

The resulting self-consistent estimate for $U^{eff}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right)$ is

$$U^{SC}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right) = \mu\left(\varepsilon_{ij}^{0}-\varepsilon_{ij}^{P0}\right)\left(\varepsilon_{ij}^{0}-\varepsilon_{ij}^{P0}\right) + \frac{1}{3}b^{SC}\varepsilon_{ij}^{P0}\varepsilon_{ij}^{P0} \tag{4.43}$$

4.5 Calculation of the effective energy $w^{eff}\left(\varepsilon_{ij}^{0}\right)$ of the composite

The prescriptions (4.36), (4.41) and (4.43) give explicit expressions for the Hashin-Shtrikman upper bound, lower bound and self-consistent estimate for $U^{eff}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right)$, respectively. It is straightforward to optimise these expressions with respect to ε_{ij}^{P0} in order to obtain the bounds and self-consistent estimate for $w^{eff}\left(\varepsilon_{ij}^{0}\right)$, as prescribed by (3.2). The Hashin-Shtrikman upper bound on $w^{eff}\left(\varepsilon_{ij}^{0}\right)$ is given by

$$w^{HS+}\left(\varepsilon_{ij}^{0}\right)=\frac{\mu b^{HS+}}{3\mu+b^{HS+}}\varepsilon_{ij}^{0}\varepsilon_{ij}^{0} \tag{4.44}$$

The algebraic expressions for the lower bound $w^{HS-}\left(\varepsilon_{ij}^{0}\right)$ and for the self-consistent estimate $w^{SC}\left(\varepsilon_{ij}^{0}\right)$ are identical to (4.44) provided b^{HS+} is replaced by b^{HS-} and b^{SC}, respectively.

5. Nonlinear Variational Principle

5.1 Upper bound

A non-linear upper bound can be obtained for the effective potential U^{eff} as defined by (3.3). We follow the procedure outlined by Ponte-Castañeda (1991, 1992). First, introduce a *linear composite* with the same microstructure as the parent non-linear composite but with a modulus $b(\boldsymbol{x})$, such that its linear potential U_L is

$$U_L\left(\varepsilon_{ij},\varepsilon_{ij}^{P},\varepsilon_{ij,k}^{P}\right)=\mu\left(\varepsilon_{ij}-\varepsilon_{ij}^{P}\right)\left(\varepsilon_{ij}-\varepsilon_{ij}^{P}\right)+\frac{1}{2}bE_P^{\,2} \tag{5.1}$$

It is assumed that the elastic shear modulus μ and the length scale λ are uniform throughout the composite, so that only $b(\boldsymbol{x})$ is different in different phases. The definition (3.3) is re-written as

$$U^{eff}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right)=\inf_{\left(\boldsymbol{u},\boldsymbol{\varepsilon}^{P}\right)\in K}\int_{\Omega}\left[\,U_L\left(\varepsilon_{ij},\varepsilon_{ij}^{P},\varepsilon_{ij,k}^{P}\right)+U\left(\varepsilon_{ij},\varepsilon_{ij}^{P},\varepsilon_{ij,k}^{P}\right)-U_L\left(\varepsilon_{ij},\varepsilon_{ij}^{P},\varepsilon_{ij,k}^{P}\right)\right]dx \tag{5.2}$$

in which the set $K=\left\{\left(\boldsymbol{u},\boldsymbol{\varepsilon}^{P}\right):\ u_i=\varepsilon_{ij}^{0}x_j \text{ on S},\ \left\langle\varepsilon_{ij}^{P}\right\rangle=\varepsilon_{ij}^{P0}\right\}$ has already been defined in (3.4). Now partition the right hand side of (5.2) to give

$$U^{eff}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right) \le \inf_{\left(\boldsymbol{u},\boldsymbol{\varepsilon}^{P}\right)\in K} \int_{\Omega}\left[U_L\left(\varepsilon_{ij},\varepsilon_{ij}^{P},\varepsilon_{ij,k}^{P}\right)\right]\,dx + \sup_{\left(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^{P},\boldsymbol{\gamma}\right)} \int_{\Omega}\left[U\left(\varepsilon_{ij},\varepsilon_{ij}^{P},\gamma_{ijk}\right)-U_L\left(\varepsilon_{ij},\varepsilon_{ij}^{P},\gamma_{ijk}\right)\right]\,dx \tag{5.3}$$

and note that the first term on the right hand side of (5.3) is $U_L^{eff}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right)$. On recalling that $U_L^{eff}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right)$ is less than its Hashin-Shtrikman upper bound $U^{HS+}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right)$ we have

$$U^{eff}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right) \le U^{HS+}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right) + \sup_{\left(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^{P},\boldsymbol{\gamma}\right)} \int_{\Omega}\left[U\left(\varepsilon_{ij},\varepsilon_{ij}^{P},\gamma_{ijk}\right)-U_L\left(\varepsilon_{ij},\varepsilon_{ij}^{P},\gamma_{ijk}\right)\right]\,dx \tag{5.4}$$

for any assumed distribution of modulus $b(\boldsymbol{x})$. We shall assume that $b(\boldsymbol{x})$ equals $b^{(r)}$ within each phase *r*, and (5.4) then reduces to

$$U^{eff}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right) \le \inf_{b^{(r)}} \left\{ U^{HS+}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right) + \sum_{r=1}^{M} p_r \sup_{\left(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^{P},\boldsymbol{\gamma}\right)} \left[U-U_L\right] \right\} \tag{5.5}$$

In order to evaluate (5.5) the Hashin-Shtrikman upper bound $U^{HS+}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right)$ is expressed by (4.36), with $b^{HS+} \equiv b^{HS+}\left(b^{(r)}\right)$ given by (4.34). The algebraic form of the second term on the right-hand side of (5.5) depends upon the choice adopted for $U\left(\varepsilon_{ij},\varepsilon_{ij}^{P},\gamma_{ijk}\right)$. Here, we consider the power law case as specified by (3.9), and routine manipulation leads to

$$U^{eff}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right) \le \inf_{b^{(r)}} \left\{ \begin{aligned} &\mu\left(\varepsilon_{ij}^{0}-\varepsilon_{ij}^{P0}\right)\left(\varepsilon_{ij}^{0}-\varepsilon_{ij}^{P0}\right)+ \\ &\frac{1}{3}b^{HS+}\varepsilon_{ij}^{P0}\varepsilon_{ij}^{P0} + \sum_{r=1}^{M} p_r\left(\frac{\sigma_r e_0}{N+1}\left(\frac{E_{Pr}}{e_0}\right)^{N+1} - \frac{1}{2}b^{(r)}E_{Pr}^{\ 2}\right) \end{aligned} \right\} \tag{5.6}$$

where

$$E_{Pr} = \left(\frac{\sigma_r}{b^{(r)}}\right)^{\frac{1}{1-N}} e_0^{\ N/(N-1)} \tag{5.7}$$

The corresponding upper bound for $w^{eff}\left(\varepsilon_{ij}^{0}\right)$ follows by substituting (5.6) into (3.2), and taking the infimum with respect to ε_{ij}^{P0}, to obtain

$$\varepsilon_{ij}^{P0} = \frac{3\mu}{3\mu + b^{HS+}}\varepsilon_{ij}^{0} \tag{5.8}$$

and thereby

$$w^{eff} \le w^{+} \equiv \inf_{b^{(r)}} \left\{ \frac{\mu b^{HS+}}{3\mu + b^{HS+}} \varepsilon_{ij}^{0} \varepsilon_{ij}^{0} + \sum_{r=1}^{M} p_r \left(\frac{\sigma_r e_0}{N+1} \left(\frac{E_{Pr}}{e_0} \right)^{N+1} - \frac{1}{2} b^{(r)} E_{Pr}{}^{2} \right) \right\} \tag{5.9}$$

with E_{Pr} given by (5.7).

The minimisation procedure in (5.9) is conducted numerically in order to obtain the optimal values of $b^{(r)}$ for a given choice of ε_{ij}^{0}. In order to perform this minimisation, use is made of the expressions (4.34) for $b^{HS+}\left(b^{(r)}\right)$ and of (4.37) for μ_Q. With the optimal choice, the estimate for the average stress within the composite $\left\langle \sigma'_{ij} \right\rangle^{+}$, associated with the Hashin-Shtrikman upper bound, is given by

$$\left\langle \sigma'_{ij} \right\rangle^{+} = \frac{\partial w^{+}}{\partial \varepsilon_{ij}^{0}} = \frac{2\mu b^{HS+}}{3\mu + b^{HS+}} \varepsilon_{ij}^{0} \tag{5.10}$$

Analytical expression for the upper bound

It is of practical usefulness to obtain an analytic expression for $w^{+}\left(\varepsilon_{ij}^{0}\right)$. In this section, an analytic expression is obtained for an upper bound on $w^{+}\left(\varepsilon_{ij}^{0}\right)$. It is shown subsequently (section 6) that this expression is highly accurate once the macroscopic plastic strain exceeds the yield strain for the softer phase. We now show that the expression (5.6) satisfies the inequality

$$U^{eff}\left(\varepsilon_{ij}^{0}, \varepsilon_{ij}^{P0}\right) < U^{+}\left(\varepsilon_{ij}^{0}, \varepsilon_{ij}^{P0}\right) \tag{5.11}$$

where $U^{+}\left(\varepsilon_{ij}^{0}, \varepsilon_{ij}^{P0}\right)$ has the algebraic form

$$U^{+}\left(\varepsilon_{ij}^{0}, \varepsilon_{ij}^{P0}\right) \equiv \mu\left(\varepsilon_{ij}^{0} - \varepsilon_{ij}^{P0}\right)\left(\varepsilon_{ij}^{0} - \varepsilon_{ij}^{P0}\right) + \frac{\sigma_{+} e_0}{N+1} \left(\frac{\varepsilon_{e}^{P0}}{e_0} \right)^{N+1} \tag{5.12}$$

for a composite with constituent phases specified by (3.9). It is noted in passing that this algebraic structure is identical to that given in (3.10) for the Voigt bound and in (3.25) for the Reuss bound. An analytic expression for $w^{+}\left(\varepsilon_{ij}^{0}\right)$ follows by minimising $U^{+}\left(\varepsilon_{ij}^{0}, \varepsilon_{ij}^{P0}\right)$ with respect to ε_{ij}^{P0}. It remains to obtain an expression for the reference strength σ_{+} of the composite.

First, relation (5.6) is simplified via (5.7) to give

$$U^{eff}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right) \le \inf_{b^{(r)}} \left\{ \begin{array}{l} \mu\left(\varepsilon_{ij}^{0}-\varepsilon_{ij}^{P0}\right)\left(\varepsilon_{ij}^{0}-\varepsilon_{ij}^{P0}\right)+ \\ \frac{1}{3}b^{HS+}\varepsilon_{ij}^{P0}\varepsilon_{ij}^{P0}+\sum_{r=1}^{M}\left(\frac{(1-N)}{2(1+N)}p_r\sigma_r e_0\left(\frac{\sigma_r}{e_0 b^{(r)}}\right)^{\frac{1+N}{1-N}}\right) \end{array} \right\} \tag{5.13}$$

with b^{HS+} specified by the limiting value of (4.34) as $b_0 \to b^{(2)}$,

$$b^{HS+} = b^{(2)}\left[1+\frac{p_1}{\frac{1}{\gamma-1}+\frac{4}{3}\left(b^{(2)}\mu_Q\right)p_2}\right] \tag{5.14}$$

and $\gamma \equiv b^{(1)}/b^{(2)}$. Now μ_Q is defined in (C10) and, with $b_0 = b^{(2)}$, can be split into a sum of two parts,

$$b^{(2)}\mu_Q = \psi_L + \psi_{R+} \tag{5.15}$$

where

$$\psi_L \equiv \frac{3}{10\lambda^2}\int_0^{\infty}\left[rh(r)e^{-r/\lambda}\right]dr \tag{5.16}$$

and the remainder term is

$$\psi_{R+} \equiv \frac{9}{20\lambda^2}\int_0^{\infty}\left[rh(r)e^{-\frac{r}{\lambda}\sqrt{1+\left(3\mu/b^{(2)}\right)}}\right]dr \tag{5.17}$$

We seek to simplify (5.13) by first carrying out a minimisation with respect to $b^{(2)}$, with $\gamma \equiv b^{(1)}/b^{(2)}$ held fixed, and then minimising with respect to γ. (This approach follows that developed by Ponte Castañeda and De Botton (1992), Suquet (1993) and Smyshlyaev and Fleck (1995) in a similar context). Unfortunately, the expression (5.14) for b^{HS+} is not homogeneous of degree 1 in $b^{(2)}$ due to the term ψ_{R+} in the composition of b^{HS+}. But, deep in the plastic range, the optimal value of $b^{(2)}$ is of the order of the secant modulus and is much smaller than μ. Hence, the contribution of ψ_{R+} to b^{HS+} is small, and b^{HS+} *can be bounded* accurately by the term ψ_L which is homogeneous of degree 1 in $b^{(2)}$. Thus we find

$$b^{(2)}\mu_Q > \psi_L \tag{5.18}$$

always, with $b^{(2)}\mu_Q$ tending to its limit ψ_L deep in the plastic range. On replacing $b^{(2)}\mu_Q$ by ψ_L in (5.14), the expression (5.14) for b^{HS+} is replaced by its bound b_L^{HS+} where

$$b^{HS+} < b_L^{HS+} \equiv b^{(2)}\left[1+\frac{p_1}{\frac{1}{\gamma-1}+\frac{4}{3}\psi_L p_2}\right] \tag{5.19}$$

Similarly, (5.13) is replaced by (5.11) where

$$U^+\left(\varepsilon_{ij}^0,\varepsilon_{ij}^{P0}\right) = \inf_{\gamma}\ \inf_{b^{(2)}} \left\{ \begin{array}{l} \mu\left(\varepsilon_{ij}^0-\varepsilon_{ij}^{P0}\right)\left(\varepsilon_{ij}^0-\varepsilon_{ij}^{P0}\right)+\frac{1}{3}b_L^{HS+}\varepsilon_{ij}^{P0}\varepsilon_{ij}^{P0}+ \\ \frac{(1-N)}{2(1+N)}e_0\left(p_1\sigma_1\left(\frac{\sigma_1}{e_0\gamma b^{(2)}}\right)^{\frac{1+N}{1-N}}+p_2\sigma_2\left(\frac{\sigma_2}{e_0 b^{(2)}}\right)^{\frac{1+N}{1-N}}\right) \end{array} \right\} \tag{5.20}$$

Since b_L^{HS+} is homogeneous of degree 1 in $b^{(2)}$, the minimisation of (5.20) with respect to $b^{(2)}$ can be conducted analytically to obtain the result (5.12), where

$$\sigma_+ \equiv \inf_{0\le\gamma\le 1}\left\{\left[1+\frac{p_1}{\frac{1}{\gamma-1}+\frac{4}{3}\psi_L p_2}\right]^{\frac{N+1}{2}}\left[p_1\gamma^{\frac{N+1}{N-1}}\sigma_1^{2/(1-N)}+p_2\sigma_2^{2/(1-N)}\right]^{\frac{1-N}{2}}\right\} \tag{5.21}$$

Further reduction can be performed in the limit of an elastic-ideally plastic solid (N=0). Then, (5.21) becomes,

$$\frac{\sigma_+}{\sigma_1} = \frac{p_1}{1-\frac{4}{3}\psi_L p_2} + p_2\sqrt{\frac{1-\frac{4}{3}\psi_L}{1-\frac{4}{3}\psi_L p_2}\left(\frac{\sigma_2}{\sigma_1}\right)^2 - \frac{\left(1-\frac{4}{3}\psi_L\right)\frac{4}{3}\psi_L p_1}{\left(1-\frac{4}{3}\psi_L p_2\right)^2}} \tag{5.22}$$

for $\sigma_2 > \sigma_1$. It remains to evaluate ψ_L for any given correlation function $h(r)$. For the choice (4.25), evaluation of (5.16) gives

$$\frac{4}{3}\psi_L = \frac{2}{5}\left(1+\frac{\lambda}{a}\right)^{-2} \tag{5.23}$$

The above derivation for σ_+ in (5.21) closely follows that outlined by Smyshlyaev and Fleck (1995) for a rigid-power law strain gradient composite. In fact, the result (5.21) is identical to that of (4.5) in Smyshlyaev and Fleck (1995), upon making the identification $\frac{4}{3}\psi_L$ with their correlation function ψ_2^+. It is recalled that Smyshlyaev and Fleck (1995) consider a fundamentally different mathematical framework for the constitutive law: in the present study the plastic strain ε_{ij}^P is treated as an independent internal variable on a similar kinematic footing to that of the displacement field. However, deep in the plastic range, the contribution from elastic strain becomes unimportant and hence it is to be expected physically that the two formulations should yield the same results. Further, Smyshlyaev and Fleck (1995) assume that the strain energy density within each phase is a power law function of the equivalent strain quantity

$$E_{SF} = \left(\left(\varepsilon_e^P \right)^2 + \frac{2}{3} \lambda_{SF}{}^2 \chi_{ij}^S \chi_{ij}^S \right)^{1/2} \tag{5.24}$$

where χ_{ij}^S is the symmetric part of the material curvature tensor,

$$\chi_{ij}^S = \frac{1}{2} e_{iqr} u_{r,jq} + \frac{1}{2} e_{jqr} u_{i,jq} \tag{5.25}$$

and the length scale λ_{SF} is to be distinguished from that of λ in the present study. With the choice (4.25) for $h(r)$, Smyshlyaev and Fleck (1995) obtain

$$\psi_2^+ = \frac{2}{5} \left(1 + \frac{\lambda_{SF}}{2a} \right)^{-2} \tag{5.26}$$

Thus, the upper bound calculation of Smyshlyaev and Fleck (1995) coincides with that derived here upon making the identification $\lambda_{SF} = 2\lambda$. In the conventional limit $\lambda_{SF} = 2\lambda \to 0$, we obtain $\frac{4}{3}\psi_L \to \frac{2}{5}$ and (5.22) gives the same upper bound measure of the uniaxial yield strength for an ideally plastic composite as that of equation (15) in Ponte Castañeda and De Botton (1992), and of equation (3.16) in Suquet (1993).

5.2 Lower estimate and self-consistent estimate

A similar procedure to obtain strict Hashin-Shtrikman lower bounds for nonlinear composites is not available. Instead, *estimates* can be derived based on the linear Hashin-Shtrikman lower bounds and self-consistent estimates in the manner suggested by Ponte-Castañeda and De Botton (1992). The idea is to evaluate (5.9), with b^{HS+} replaced by the lower bound b^{HS-} to produce a 'lower estimate' w^- for $w^{eff}\left(\varepsilon_{ij}^0\right)$; in like manner, b^{HS+} replaced by its self-consistent estimate b^{SC} produces a 'self-consistent estimate' w^{SC} for $w^{eff}\left(\varepsilon_{ij}^0\right)$.

Analytical expression for the lower estimate

An analytical approximation $U^{-}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right)$ for the lower estimate of $U^{eff}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right)$ can be obtained in a similar manner to that described above for the upper bound[3] $U^{+}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right)$. We shall show that $U^{-}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right)$ can be written in the form

$$U^{-}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right)\equiv \mu\left(\varepsilon_{ij}^{0}-\varepsilon_{ij}^{P0}\right)\left(\varepsilon_{ij}^{0}-\varepsilon_{ij}^{P0}\right)+\frac{\sigma_{-}e_{0}}{N+1}\left(\frac{\varepsilon_{e}^{P0}}{e_{0}}\right)^{N+1} \tag{5.27}$$

and we shall obtain an expression for the reference strength σ_{-}.

The lower estimate for $U^{eff}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right)$ follows from (5.13) as

$$U_{LE}^{eff}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right)=\inf_{b^{(r)}}\left\{\begin{array}{l}\mu\left(\varepsilon_{ij}^{0}-\varepsilon_{ij}^{P0}\right)\left(\varepsilon_{ij}^{0}-\varepsilon_{ij}^{P0}\right)+\\ \frac{1}{3}b^{HS-}\varepsilon_{ij}^{P0}\varepsilon_{ij}^{P0}+\sum_{r=1}^{M}\left(\frac{(1-N)}{2(1+N)}p_{r}\sigma_{r}e_{0}\left(\frac{\sigma_{r}}{e_{0}b^{(r)}}\right)^{\frac{1+N}{1-N}}\right)\end{array}\right\} \tag{5.28}$$

where (4.34) provides

$$b^{HS-}=b^{(1)}\left[1+\frac{p_{2}}{\frac{1}{\gamma-1}+\frac{4}{3}\left(b^{(1)}\mu_{Q}\right)p_{1}}\right] \tag{5.29}$$

upon taking $b_{0}\equiv b^{(1)}$ and $\gamma=b^{(2)}/b^{(1)}$. The term $b^{(1)}\mu_{Q}$ follows from (C10) as

$$b^{(1)}\mu_{Q}=\psi_{L}+\psi_{R-} \tag{5.30}$$

where ψ_{L} has already been defined in (5.16) and

$$\psi_{R-}\equiv\frac{9}{20\lambda^{2}}\int_{0}^{\infty}\left[rh(r)e^{-\frac{r}{\lambda}\sqrt{1+\left(3\mu/b^{(1)}\right)}}\right]dr \tag{5.31}$$

As for the upper bound calculation above, $b^{(1)}\mu_{Q}$ is bounded by ψ_{L}, and so b^{HS-} is bounded by b_{L}^{HS-} such that

$$b^{HS-}<b_{L}^{HS-}=b^{(1)}\left[1+\frac{p_{2}}{\frac{1}{\gamma-1}+\frac{4}{3}\psi_{L}p_{1}}\right] \tag{5.32}$$

Consequently, the lower estimate (5.28) is bounded by $U^{-}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right)$ as defined by

$$U_{LE}^{eff}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right) < U^{-}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right) \equiv \inf_{b^{(r)}} \left\{ \begin{array}{l} \mu\left(\varepsilon_{ij}^{0}-\varepsilon_{ij}^{P0}\right)\left(\varepsilon_{ij}^{0}-\varepsilon_{ij}^{P0}\right)+ \\ \frac{1}{3} b_{L}^{HS-}\varepsilon_{ij}^{P0}\varepsilon_{ij}^{P0} + \sum_{r=1}^{M}\left(\frac{(1-N)}{2(1+N)} p_r \sigma_r e_0 \left(\frac{\sigma_r}{e_0 b^{(r)}}\right)^{\frac{1+N}{1-N}}\right) \end{array} \right\} \quad (5.33)$$

The expression (5.33) for $U^{-}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right)$ can be minimised analytically with respect to $b^{(1)}$ to the desired form (5.27), where

$$\sigma_{-} \equiv \inf_{0\le\omega\le 1} \left\{ \left[1+\frac{p_2}{\frac{\omega}{1-\omega}+\frac{4}{3}\psi_L p_1} \right]^{\frac{N+1}{2}} \left[p_2 \omega^{\frac{1+N}{1-N}} \sigma_2^{\frac{2}{1-N}} + p_1 \sigma_1^{\frac{2}{1-N}} \right]^{\frac{1-N}{2}} \right\} \quad (5.34)$$

In the limit $N \to 0$, an explicit expression for the estimate of the effective uniaxial yield strength σ_{-} is available,

$$\frac{\sigma_{-}}{\sigma_1} = \frac{p_2}{1-\frac{4}{3}\psi_L p_1}\frac{\sigma_2}{\sigma_1} + \sqrt{\frac{1-\frac{4}{3}\psi_L}{1-\frac{4}{3}\psi_L p_1} - \frac{p_2 \frac{4}{3}\psi_L\left(1-\frac{4}{3}\psi_L\right)}{\left(1-\frac{4}{3}\psi_L p_1\right)^2}\left(\frac{\sigma_2}{\sigma_1}\right)^2} \quad (5.35a)$$

$$\text{if} \quad \frac{4}{3}\psi_L \sqrt{1+\frac{1-\frac{4}{3}\psi_L}{\frac{4}{3}\psi_L} p_2} < \frac{\sigma_1}{\sigma_2} < 1$$

and

$$\frac{\sigma_{-}}{\sigma_1} = \sqrt{1+p_2\left(\frac{1}{\frac{4}{3}\psi_L}-1\right)} \quad (5.35b)$$

$$\text{if} \quad \frac{\sigma_1}{\sigma_2} < \frac{4}{3}\psi_L \sqrt{1+\frac{1-\frac{4}{3}\psi_L}{\frac{4}{3}\psi_L} p_2} \; .$$

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[3] No such analytical expression exists for the self-consistent estimate of U^{eff}.

The results (5.34) and (5.35) are identical to those of (4.13) and (4.14), respectively, in Smyshlyaev and Fleck (1995), upon switching the material indices $1 \leftrightarrow 2$, and upon making the identifications $\lambda_{SF} = 2\lambda$ and $\frac{4}{3}\psi_L \equiv \psi^-$; the correlation function ψ^- is defined in Smyshlyaev and Fleck (1995). In the conventional limit $\lambda \to 0$, we obtain $\frac{4}{3}\psi_L = \psi^- \to \frac{2}{5}$ and (5.35) reduces to the lower estimate of Ponte Castañeda and De Botton (1992), formula (16).

6. Numerical results

A selected set of numerical calculations has been performed in order to explore the dependence of macroscopic strength upon the ratio of correlation length *a* to microstructural length scale λ for the choice *N*=0.3, σ_1/μ=1/100, σ_2/μ=3/100, p_2=0.3, e_0=1. Elementary and Hashin-Shtrikman bounds and estimates are shown in Figure 1 for the macroscopic uniaxial stress versus strain curve σ/σ_1 versus $\mu\varepsilon/\sigma_1$, with the choices λ/*a*= 0.01, 0.1 and 1. Whilst the Reuss and Voigt bounds are independent of λ/*a* the Hashin-Shtrikman bounds and estimates for the stress-strain curves increase towards the Voigt upper bound with increasing λ/a. For each value of λ/a, the self-consistent estimate is approximately mid-way between the Hashin-Shtrikman upper bound and lower estimate. The Hashin-Shtrikman bounds and estimates required numerical calculations based on the formula (5.13), and its 'lower estimate' and 'self-consistent' modifications.

Recall that the macroscopic stress versus plastic strain response of the composite is power law in nature for both the Voigt and Reuss bounds, as stated by (3.10) and (3.25). Further, the non-linear Hashin-Shtrikman upper bound and lower estimate tend to the power law relations (5.12) and (5.27) deep in the plastic range. The macroscopic stress versus plastic strain response of the composite is plotted in Figure 2 using log-log axes, for the choice λ/*a*= 0.1, *N*=0.3, σ_1/μ=1/100, σ_2/μ=3/100, p_2=0.3, e_0=1. The power law approximation (5.12) to the Hashin-Shtrikman upper bound is shown as a dashed line, and it is clear that the full elastic-plastic composite response is adequately represented by (5.12) provided $\mu\varepsilon_e^{P0}/\sigma_1$ exceeds a value of about unity. Thus, the power law approximation is accurate provided the macroscopic strain is comparable to or exceeds the yield strain $\sigma_1/3\mu$ of the softer phase. In similar fashion, the Hashin-Shtrikman lower estimate and self-consistent estimate are closely approximated by power-law fits provided $\mu\varepsilon_e^{P0}/\sigma_1$ exceeds a value of about unity. The power law relation for the self-consistent estimate reads

$$U^{eff} \approx U^{SC}\left(\varepsilon_{ij}^0, \varepsilon_{ij}^{P0}\right) \equiv \mu\left(\varepsilon_{ij}^0 - \varepsilon_{ij}^{P0}\right)\left(\varepsilon_{ij}^0 - \varepsilon_{ij}^{P0}\right) + \frac{\sigma_{SC} e_0}{N+1}\left(\frac{\varepsilon_e^{P0}}{e_0}\right)^{N+1} \tag{6.1}$$

so that, in uniaxial tension, the uniaxial tensile stress σ on the composite is related to the uniaxial plastic strain ε_e^{P0} by

$$\sigma = \sigma_{SC}\left(\frac{\varepsilon_e^{P0}}{e_0}\right)^N \tag{6.2}$$

The magnitude of the strength parameter σ_{SC} is obtained by fitting (6.2) to the stress versus plastic strain curve at a sufficiently large value of ε_e^{P0} - we adopt the value $\mu\varepsilon_e^{P0}/\sigma_1$=20. The strength measure σ_+ for the Hashin-Shtrikman upper bound and σ_- for the lower estimate are obtained via (5.21), (5.22), (5.34) and (5.35), or by fitting the power law relations of type (6.2) to the numerical results at the large value of $\mu\varepsilon_e^{P0}/\sigma_1$=20; the former method was used for *N*=0 and the latter method was used for finite *N* in the current study.

It is instructive to explore in Figure 3 the dependence of the strength parameters $\Sigma = \left(\sigma_V, \sigma_R, \sigma_+, \sigma_-, \sigma_{SC}\right)$ upon the ratio of length scales λ/a at selected values of strain hardening exponent *N*=0, 0.1 and 0.3. As before, we choose σ_1/μ=1/100, σ_2/μ=3/100, p_2=0.3, e_0=1 for illustrative purposes. It is clear from the figure that the elementary bounds, σ_V for Voigt and σ_R for Reuss, are insensitive to the value of λ/a, whilst the more sophisticated upper bound σ_+, and estimates σ_- and σ_{SC}, increase towards the Voigt limit with increasing λ/a. For the choice λ/a=0.3, the intermediate strengths $\left(\sigma_+, \sigma_-, \sigma_{SC}\right)$ lay approximately mid-way between their limits at large and small λ/a. We further note from Fig. 3 that the sensitivity of the strength to *N* increases in the order $\left(\sigma_V, \sigma_+, \sigma_{SC}, \sigma_-, \sigma_R\right)$ but is only significant for the Hashin-Shtrikman lower estimate and for the Reuss bound.

The effect of volume fraction p_2 of the stronger phase upon the macroscopic strength parameters σ_V, σ_R and σ_{SC} is plotted in Figure 4, for the choice *N*=0.3, σ_1/μ=1/100, σ_2/μ=3/100, e_0=1, and λ/a=0 and 0.3. The results for the Voigt and Reuss bounds and for the conventional limit (λ/a=0) of the self-consistent composite are well-known: as p_2 increases the composite strength increases, with the self-consistent prediction moving closer to the Voigt prediction. It is evident that the self-consistent estimate at λ/a=0.3 lies about mid-way between the self-consistent estimate for λ/a=0 and that of the Voigt upper bound. These self-consistent calculations reveal that the size effect is of greatest significance at small values of p_2; a similar conclusion can be drawn for the Hashin-Shtrikman upper bound and lower estimate, but further plots are omitted for the sake of brevity.

7. Concluding remarks

The analysis presented above has a number of novel features, as follows.

(i) The recent Fleck-Hutchinson deformation theory version of strain gradient plasticity, (Fleck and Hutchinson, 2001) has been modified for simpler analytical and numerical exploitation such that the plastic strain tensor is treated as an internal variable on an equal footing to material displacement. The elastic strain contribution to the strain energy resembles a penalty function, such that deep in the plastic range the plastic strain is (almost) compatible with the material displacement field.

(ii) The homogenisation procedure given above deals with the internal state variables. Deep in the plastic range, the relation between the macroscopic effective stress and plastic strain is closely related to that for a rigid-plastic solid, despite the widely different underlying mathematical formulations.

(iii) The size effects determined in the present study for an elastic-plastic composite are slightly stronger than those found previously by Smyshlyaev and Fleck (1995) for a rigid-plastic composite. This may be rationalised as follows. Smyshlyaev and Fleck (1995) considered a reduced couple stress version of strain gradient plasticity wherein the gradient term was limited to the symmetric part of the material curvature, as expressed by (5.24) and (5.25). In contrast, the strain gradient measure (2.6) involves all terms of the plastic strain gradient. The precise details on the contribution from the various strain gradient terms to the overall size effect will be explored in a subsequent study.

(iv) The effect of strain gradient strengthening is to force the strain distribution within a composite to become spatially more uniform. Consequently, the bounds and estimates for the composite tend to the Voigt upper bound as the microstructure becomes fine in relation to the internal material length scale, see Figure 3 for example.

(v) The present formulation assumes bonded phases such that both the displacement field and the plastic strain field are continuous across all interfaces. In reality, interfaces can impede slip. A natural next step might be to allow for this effect by the admission of a thin interphase layer analogous to that introduced in the context of elasticity by Benveniste and Miloh (2001) and Hashin (2002).

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Appendix A: Calculation of $V^{eff}\left(\varepsilon_{ij}^{P0}\right)$

The purpose of this Appendix is to simplify the expression (3.3) for $U^{eff}\left(\varepsilon_{ij}^{0}, \varepsilon_{ij}^{P0}\right)$ when the potential $U\left(\varepsilon_{ij}, \varepsilon_{ij}^{P}, \varepsilon_{ij,k}^{P}\right)$ takes the form (2.5b). Suffixes will henceforth be suppressed, to simplify the notation. First, fix ε^{P} and calculate the infimum in (3.3) over all admissible fields ε. This infimum is attained when

$$\int_{\Omega} \delta\varepsilon\, L\left(\varepsilon - \varepsilon^{P}\right) dx = 0 \qquad \text{(A1)}$$

for all allowed variations $\delta\varepsilon$: these are associated with displacement fields δu that vanish on the boundary S. The strain field that attains the infimum is

$$\varepsilon = \varepsilon^{0} + \tilde{\varepsilon}, \qquad \text{(A2)}$$

where

$$\tilde{\varepsilon} = \Gamma L \varepsilon^{P}. \qquad \text{(A3)}$$

The operator Γ is related to the Green's function for a medium with an elastic modulus L, see Willis (1977). It is developed explicitly for an incompressible isotropic solid of shear modulus μ in Appendix B.

Then,

$$\begin{aligned}\int_{\Omega} \tfrac{1}{2}\left(\varepsilon - \varepsilon^{P}\right) L \left(\varepsilon - \varepsilon^{P}\right) dx &= \int_{\Omega} \tfrac{1}{2}\Big[\left(\varepsilon^{0} - \varepsilon^{P}\right) L \left(\varepsilon^{0} - \varepsilon^{P}\right) \\ &\quad + \left(\varepsilon^{0} - \varepsilon^{P}\right) L \tilde{\varepsilon} + \tilde{\varepsilon} L \left(\varepsilon - \varepsilon^{P}\right)\Big] dx \\ &= \int_{\Omega} \tfrac{1}{2}\Big[\left(\varepsilon^{0} - \varepsilon^{P}\right) L \left(\varepsilon^{0} - \varepsilon^{P}\right) - \varepsilon^{P} L \tilde{\varepsilon}\ \Big]\ dx,\end{aligned} \qquad \text{(A4)}$$

since $\tilde{\varepsilon}$ is an admissible variation $\delta\varepsilon$ (with mean value zero).

Second, evaluate the infimum with respect to plastic strains ε^{P}, with prescribed mean value ε^{P0}. This gives the result

$$U^{eff}\left(\varepsilon^{0}, \varepsilon^{P0}\right) = \tfrac{1}{2}\left(\varepsilon^{0} - \varepsilon^{P0}\right) L \left(\varepsilon^{0} - \varepsilon^{P0}\right) + V^{eff}\left(\varepsilon^{P0}\right), \qquad \text{(A5)}$$

where

$$V^{eff}\left(\varepsilon^{P0}\right) = \inf \int_{\Omega} \Big[\tfrac{1}{2}\left(\varepsilon^{P} - \varepsilon^{P0}\right) L \left(\varepsilon^{P} - \varepsilon^{P0}\right) - \tfrac{1}{2}\varepsilon^{P} L\, \Gamma L \varepsilon^{P} + V\left(\varepsilon^{P}, \nabla\varepsilon^{P}\right)\Big]\ dx. \qquad \text{(A6)}$$

The point of this expression is that it demonstrates explicitly that the effective potential U^{eff} can be expressed in the form (A5).

Appendix B: Derivation of the integral operator Γ.

The system of partial differential equations (4.10) in $(u_i, \varepsilon_{ij}^P)$ can be solved for a given distribution of $\alpha_{ij}(\boldsymbol{x})$. First, introduce the Green's function $G_{ip}(\boldsymbol{x},\boldsymbol{x}')$ for an incompressible and isotropic solid such that

$$\mu \frac{\partial^2 G_{ip}(\boldsymbol{x},\boldsymbol{x}')}{\partial x_j \partial x_j} + \delta_{ip}\delta(\boldsymbol{x}-\boldsymbol{x}') = P_{p,i} \tag{B1}$$

where $\delta(\boldsymbol{x}-\boldsymbol{x}')$ is the 3D Dirac delta function and P is an equilibrating pressure field. Since $G_{ip}(\boldsymbol{x},\boldsymbol{x}')$ is an incompressible field, it satisfies $\dfrac{\partial G_{ip}(\boldsymbol{x},\boldsymbol{x}')}{\partial x_i} = 0$. Multiplication of (4.10b) by $G_{ip}(\boldsymbol{x},\boldsymbol{x}')$ and integration by parts over the domain Ω provides the solution for u_i,

$$u_p(\boldsymbol{x}') = u_p^0(\boldsymbol{x}') + \int_\Omega \left[2\mu \frac{\partial G_{ip}(\boldsymbol{x},\boldsymbol{x}')}{\partial x_j} \varepsilon_{ij}^P(\boldsymbol{x}) \right] dx \tag{B2}$$

where $u_p^0(\boldsymbol{x}')$ is the displacement field for a uniform incompressible solid of shear modulus μ subjected to given the boundary data. In the present problem, we can take $u_p^0(\boldsymbol{x}') = \varepsilon_{pj}^0 x_j'$ where the uniform strain ε_{pj}^0 is specified, with $\varepsilon_{pp}^0 = 0$. Differentiation of (B2), followed by symmetrization, yields the corresponding strain field $\varepsilon_{ij}(\boldsymbol{x}')$,

$$\varepsilon_{ij}(\boldsymbol{x}') = \varepsilon_{ij}^0 + \int_\Omega \left[\Gamma_{ijkl}(\boldsymbol{x},\boldsymbol{x}') 2\mu \left(\varepsilon_{kl}^P(\boldsymbol{x}) - \varepsilon_{kl}^{P0} \right) \right] dx \tag{B3}$$

where Γ has the components

$$\Gamma_{ijkl}(\boldsymbol{x},\boldsymbol{x}') = \frac{\partial^2 G_{ki}(\boldsymbol{x}',\boldsymbol{x})}{\partial x_l \partial x_j'} \qquad \text{symm } (i,j),(k,l) \tag{B4}$$

As discussed by Willis (1977), the integral operation on ε_{kl}^{P0} produces zero as Γ_{ijkl} has zero mean value over Ω. Except in a boundary layer, the presence of ε_{kl}^{P0} permits the replacement of $\Gamma_{ijkl}(\boldsymbol{x},\boldsymbol{x}')$ by its corresponding infinite body form $\Gamma_{ijkl}(\boldsymbol{x}-\boldsymbol{x}')$, and permits integration over Ω to be replaced by integration over all space, in the asymptotic limit of fine microstructure.

Now proceed to solve (4.10a) for $\left(\varepsilon_{ij}^P - \varepsilon_{ij}^{P0}\right)$ in terms of a given $\left(\alpha_{ij} - \langle \alpha_{ij} \rangle\right)$. Since (4.10a) is of Helmholtz type, it is convenient to introduce the scalar Green's function $g(\boldsymbol{x})$ that satisfies

$$\nabla^2 g(\boldsymbol{x}) - k^2 g(\boldsymbol{x}) + \delta(\boldsymbol{x}) = 0 \tag{B5}$$

where we have made the identification

$$k^2 = \frac{3\mu + b_0}{b_0 \lambda^2} \tag{B6}$$

The explicit solution of (B5) is

$$g(\boldsymbol{x}) = e^{-kr} / 4\pi r \tag{B7}$$

where $r = |\boldsymbol{x}|$. A comparison of (4.10a) with (B5) provides the solution of (4.10a) by superposition,

$$\varepsilon_{ij}^{P}(\boldsymbol{x}) = \varepsilon_{ij}^{P0} + \frac{3}{2 b_0 \lambda^2} \int \left[g(\boldsymbol{x} - \boldsymbol{x}') \left\{ 2\mu \left(\varepsilon_{ij}(\boldsymbol{x}') - \varepsilon_{ij}^{0} \right) - \left(\alpha_{ij} - \langle \alpha_{ij} \rangle \right)(\boldsymbol{x}') \right\} \right] dx' \tag{B8}$$

Upon noting that the term $\varepsilon_{ij}(\boldsymbol{x}') - \varepsilon_{ij}^{0}$ on the right hand side of (B8) can be expressed in terms of ε_{ij}^{P} via the integral expression (B3), we conclude that (B8) is an implicit expression for ε_{ij}^{P} in terms of the polarisation $\left(\alpha_{ij} - \langle \alpha_{ij} \rangle \right)$. In order to obtain the connection directly, we take Fourier Transforms of each of the terms in (B3) and (B8), and apply the Convolution Theorem. Define the 3D Fourier Transform of an arbitrary function $f(\boldsymbol{x})$ by

$$\hat{f}(\xi) = \int \left[e^{i\xi.x} f(\boldsymbol{x}) \right] dx \tag{B9a}$$

with inverse

$$f(\boldsymbol{x}) = \frac{1}{8\pi^3} \int \left[e^{-i\xi.x} \hat{f}(\boldsymbol{\xi}) \right] d\xi \tag{B9b}$$

Then, application of the Convolution Theorem to (B8) gives

$$\hat{\varepsilon}_{ij}^{P} - \hat{\varepsilon}_{ij}^{P0} = \frac{3\hat{g}}{2 b_0 \lambda^2} \left\{ 4\mu^2 \hat{\Gamma}_{ijkl} \left(\hat{\varepsilon}_{kl}^{P} - \hat{\varepsilon}_{kl}^{P0} \right) - \left(\hat{\alpha}_{ij} - \langle \hat{\alpha}_{ij} \rangle \right) \right\} \tag{B10}$$

Upon noting the identity $2\mu \hat{\Gamma}\, \hat{g}\, \hat{\Gamma} = \hat{g}\, \hat{\Gamma}$, the relation (B10) can be rearranged to the form

$$\hat{\varepsilon}_{ij}^{P} - \hat{\varepsilon}_{ij}^{P0} = \frac{9\mu^2 \hat{g}^2}{b_0 \lambda^2 \left(3\mu \hat{g} - b_0 \lambda^2 \right)} \hat{\Gamma}_{ijkl} \left(\hat{\alpha}_{kl} - \langle \hat{\alpha}_{kl} \rangle \right) - \frac{3\hat{g}}{2 b_0 \lambda^2} \left(\hat{\alpha}_{ij} - \langle \hat{\alpha}_{ij} \rangle \right) \tag{B11}$$

Further reduction requires explicit expressions for $\hat{g}$ and for $\hat{\Gamma}$. Upon taking Fourier transforms of (B1), (B4) and (B5) we have

$$\hat{g} = \frac{b_0 \lambda^2}{3\mu + b_0 + b_0 \lambda^2 |\boldsymbol{\xi}|^2} \tag{B12}$$

and

$$\hat{\Gamma}_{ijkl} = \frac{1}{\mu}\left(\frac{\delta_{ik}\xi_j\xi_l}{|\boldsymbol{\xi}|^2} - \frac{\xi_i\xi_j\xi_k\xi_l}{|\boldsymbol{\xi}|^4}\right) \qquad \text{, symm } (i,j),(k,l) \tag{B13}$$

Thus, equation (B11) can be re-written in the form

$$\hat{\varepsilon}_{ij}^P - \hat{\varepsilon}_{ij}^{P0} = -\hat{\Lambda}_{ijkl}\left(\hat{\alpha}_{kl} - \langle\hat{\alpha}_{kl}\rangle\right) \tag{B14}$$

where

$$\hat{\Lambda}_{ijkl}(\boldsymbol{\xi}) = \frac{3I'_{ijkl}}{6\mu + 2b_0\left(1 + \lambda^2|\boldsymbol{\xi}|^2\right)} + \frac{9\mu}{b_0\left(1 + \lambda^2|\boldsymbol{\xi}|^2\right)\left(3\mu + b_0\left(1 + \lambda^2|\boldsymbol{\xi}|^2\right)\right)}\left(\frac{\delta_{ik}\xi_j\xi_l}{|\boldsymbol{\xi}|^2} - \frac{\xi_i\xi_j\xi_k\xi_l}{|\boldsymbol{\xi}|^4}\right)$$

$$\text{, symm } (i,j),(k,l) \tag{B15}$$

in terms of the incompressible identity tensor I'_{ijkl},

$$I'_{ijkl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}\right) \tag{B16}$$

Recall that $\Lambda_{ijkl}(\boldsymbol{x})$ is the desired integral operator relating $\left(\varepsilon_{ij}^P - \varepsilon_{ij}^{P0}\right)$ to $\left(\alpha_{ij} - \langle\alpha_{ij}\rangle\right)$ in (4.16). The statement (B15) is an explicit expression for the Fourier transform of $\Lambda_{ijkl}(\boldsymbol{x})$.

Appendix C. Evaluation of the shear compliance quantity μ_Q

In order to evaluate the expression (4.35) for the effective energy $U^{eff}\left(\varepsilon_{ij}^{0},\varepsilon_{ij}^{P0}\right)$ it is necessary to calculate b^{eff} via (4.34), but this requires the determination of μ_Q. Recall that μ_Q is defined by (4.32), and so we need to consider Q_{ijkl} in some detail. We begin by re-stating the definition (4.26) of Q_{ijkl} as

$$Q_{ijkl} = \int\left[\Lambda_{ijkl}(\boldsymbol{x})h(r)\right]dx = \frac{1}{8\pi^3}\int d\boldsymbol{x}\int d\boldsymbol{\xi}\left[e^{-i\xi.x}\hat{\Lambda}_{ijkl}(\boldsymbol{\xi})h(r)\right] \quad \text{(C1)}$$

in terms of the Fourier transform $\hat{\Lambda}_{ijkl}(\boldsymbol{\xi})$, as given explicitly in (B15). The contraction Q_{ijij} provides the value $10\mu_Q$ according to (4.32), and so (C1) and (B15) are contracted to read

$$10\mu_Q = \frac{1}{8\pi^3}\int dx\int d\boldsymbol{\xi}\left(e^{-i\xi.x}h(r)\left[\frac{9}{6\mu+2b_0\left(1+\rho^2\lambda^2\right)}+\frac{3}{b_0\left(1+\rho^2\lambda^2\right)}\right]\right) \quad \text{(C2)}$$

where $\rho \equiv |\boldsymbol{\xi}|$. The integral (C2) is evaluated by factorising the volume integral with respect to $\boldsymbol{\xi}$ into a radial part and a spherical part. Upon writing $\boldsymbol{\xi} \equiv \rho\boldsymbol{\eta}$ where $\boldsymbol{\eta} = \boldsymbol{\xi}/|\boldsymbol{\xi}|$, (C2) can be re-phrased as

$$\mu_Q = \frac{1}{80\pi^3}\int dx\, h(r)\int_{|\eta|=1} d\boldsymbol{\eta}\int_0^\infty d\rho\left(\frac{3e^{-i(\xi.x)}}{2b_0\lambda^2}\left[5-\frac{3k_1^{\,2}}{\rho^2+k_1^{\,2}}-\frac{2k_2^{\,2}}{\rho^2+k_2^{\,2}}\right]\right) \quad \text{(C3)}$$

where the constants k_1 and k_2 are defined by

$$k_1^{\,2} = \frac{3\mu+b_0}{b_0\lambda^2} \qquad \text{and} \qquad k_2^{\,2} = \frac{1}{\lambda^2} \quad \text{(C4)}$$

Now perform the integral with respect to ρ. For any well-behaved even function $a(\boldsymbol{\eta})$ of $\boldsymbol{\eta}$,

$$\int_{|\eta|=1} d\boldsymbol{\eta}\int_0^\infty d\rho\left[a(\boldsymbol{\eta})e^{-i\rho(\eta.x)}\right] = \frac{1}{2}\int_{|\eta|=1} d\boldsymbol{\eta}\int_{-\infty}^\infty d\rho\left[a(\boldsymbol{\eta})e^{-i\rho(\eta.x)}\right] = \int_{|\eta|=1} d\boldsymbol{\eta}\left[\pi\delta(\boldsymbol{\eta}.\boldsymbol{x})a(\boldsymbol{\eta})\right] \quad \text{(C5)}$$

and

$$\int_{|\eta|=1} d\boldsymbol{\eta}\int_0^\infty d\rho\left[\frac{a(\boldsymbol{\eta})}{\rho^2+k^2}e^{-i\rho(\eta.x)}\right] = \frac{1}{2}\int_{|\eta|=1} d\boldsymbol{\eta}\int_{-\infty}^\infty d\rho\left[\frac{a(\boldsymbol{\eta})}{\rho^2+k^2}e^{-i\rho(\eta.x)}\right] = \int_{|\eta|=1} d\boldsymbol{\eta}\left[\frac{\pi}{2}\frac{a(\boldsymbol{\eta})}{k}e^{-k|\eta.x|}\right] \quad \text{(C6)}$$

Upon making use of (C5) and (C6), the integral (C3) for μ_Q reduces to

$$\mu_Q = \frac{3}{160\pi^2 b_0 \lambda^3} \int dx\, h(r) \int_{|\boldsymbol{\eta}|=1} d\boldsymbol{\eta} \left[5\lambda\delta(\boldsymbol{\eta}.\boldsymbol{x}) - e^{\frac{-|\boldsymbol{\eta}.\boldsymbol{x}|}{\lambda}} - \frac{3}{2}\sqrt{\frac{3\mu + b_0}{b_0}}\, e^{\frac{-|\boldsymbol{\eta}.\boldsymbol{x}|}{\lambda}\sqrt{\frac{3\mu+b_0}{b_0}}} \right] \tag{C7}$$

Further reduction is achieved by making use of the integral formulae

$$\int dx \int_{|\boldsymbol{\eta}|=1} d\boldsymbol{\eta} \left[h(r) e^{-k|\boldsymbol{\eta}.\boldsymbol{x}|} \right] = \frac{16\pi^2}{k} \int_0^\infty dr \left[rh(r) - rh(r) e^{-kr} \right] \tag{C8}$$

for an arbitrary constant *k*, and

$$\int dx \int_{|\boldsymbol{\eta}|=1} d\boldsymbol{\eta} \left[h(r) \delta(\boldsymbol{\eta}.\boldsymbol{x}) \right] = 8\pi^2 \int_0^\infty dr \left[rh(r) \right] \tag{C9}$$

to obtain

$$\mu_Q = \frac{3}{20 b_0 \lambda^2} \int_0^\infty dr \left[rh(r) \left(2e^{-\frac{r}{\lambda}} + 3e^{-\frac{r}{\lambda}\sqrt{\frac{3\mu+b_0}{b_0}}} \right) \right] \tag{C10}$$

This is the general expression for μ_Q, for any given $h(r)$. In the present study we make the particular choice (4.25) for $h(r)$, and (C10) is then given by the analytical formula (4.37).

Figure Captions

Figure 1. Macroscopic uniaxial stress-strain response for a two phase composite, with N=0.3, $\sigma_1 / \mu = 0.01$, $\sigma_2 / \mu = 0.03$, p_2=0.3, e_0=1. HS+ denotes the upper Hashin-Shtrikman bound, HS- denotes the lower Hashin-Shtrikman estimate, and SC denotes the Hashin-Shtrikman self-consistent estimate. (a) λ / a=0.01; (b) λ / a =0.1; (c) λ / a=1.0 .

Figure 2. Macroscopic uniaxial stress σ versus uniaxial plastic strain ε_e^{P0} response for a two phase composite, with λ / a=0.1, N=0.3, $\sigma_1 / \mu = 0.01$, $\sigma_2 / \mu = 0.03$, p_2=0.3, e_0=1. HS+ denotes the upper Hashin-Shtrikman bound, HS- denotes the lower Hashin-Shtrikman estimate, and SC denotes the Hashin-Shtrikman self-consistent estimate. The dotted line is the power law extrapolation of the HS+ response down to low values of plastic strain.

Figure 3. Effect of the ratio of length scales λ / a upon the macroscopic flow strength Σ for N=0, 0.1 and 0.3, p_2=0.3, $\sigma_1 / \mu = 0.01$, $\sigma_2 / \mu = 0.03$, e_0=1. For the Voigt estimate $\Sigma \equiv \sigma_V$, whilst for the Reuss estimate $\Sigma \equiv \sigma_R$. HS+ denotes the upper Hashin-Shtrikman bound with $\Sigma \equiv \sigma_+$, HS- denotes the lower Hashin-Shtrikman estimate with $\Sigma \equiv \sigma_-$, and SC denotes the Hashin-Shtrikman self-consistent estimate with $\Sigma \equiv \sigma_{SC}$.

Figure 4. Effect of the volume fraction p_2 upon the macroscopic flow strength Σ for λ / a=0 and 0.3, N=0.3, $\sigma_1 / \mu = 0.01$, $\sigma_2 / \mu = 0.03$, e_0=1. For the Voigt estimate $\Sigma \equiv \sigma_V$, whilst for the Reuss estimate $\Sigma \equiv \sigma_R$. SC denotes the Hashin-Shtrikman self-consistent estimate with $\Sigma \equiv \sigma_{SC}$.

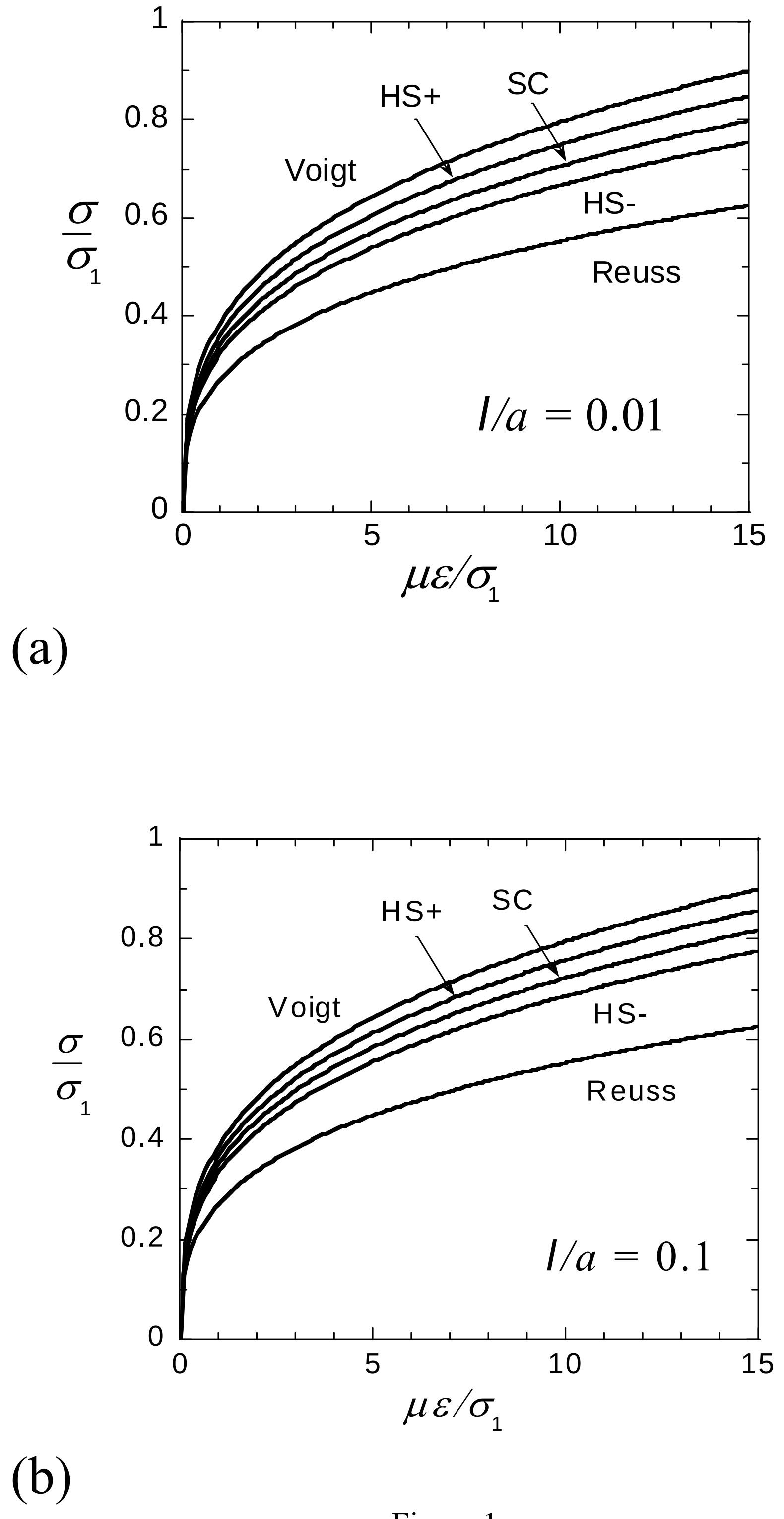
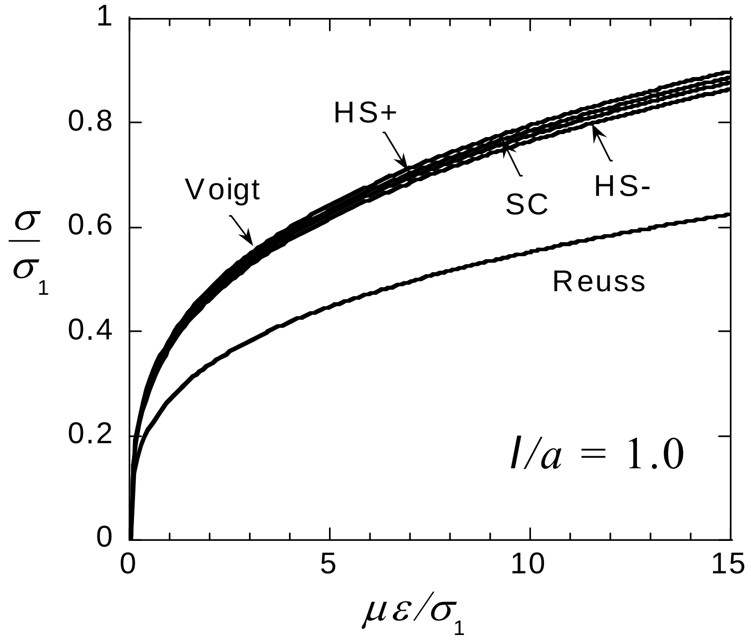

1
0.8
0.6
0.4
0.2
0
0
5
10
15
Voigt
HS+
SC
HS-
Reuss
l/a = 0.01
(a)
Voigt
HS+
SC
HS-
Reuss
l/a = 0.1
(b)

Figure 1



(c)

Figure 1

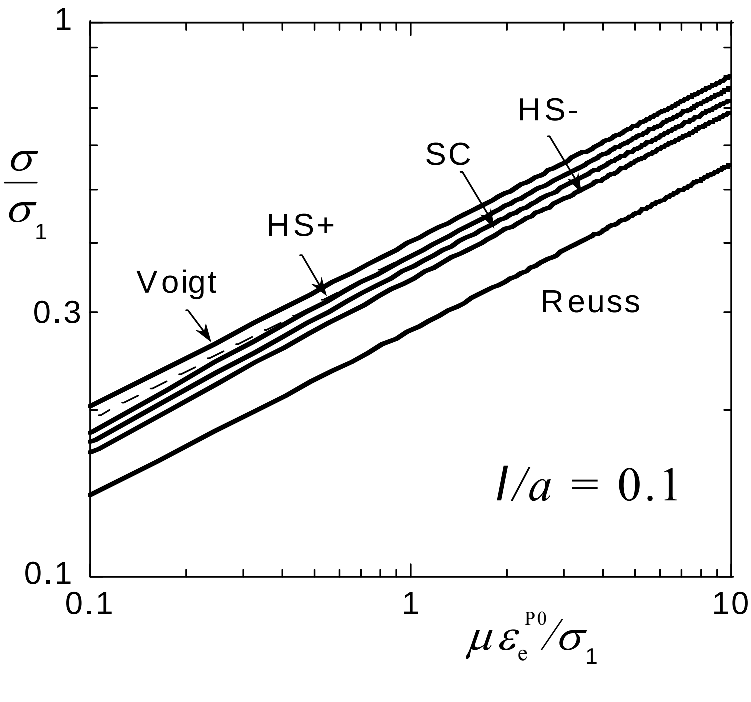


Figure 2

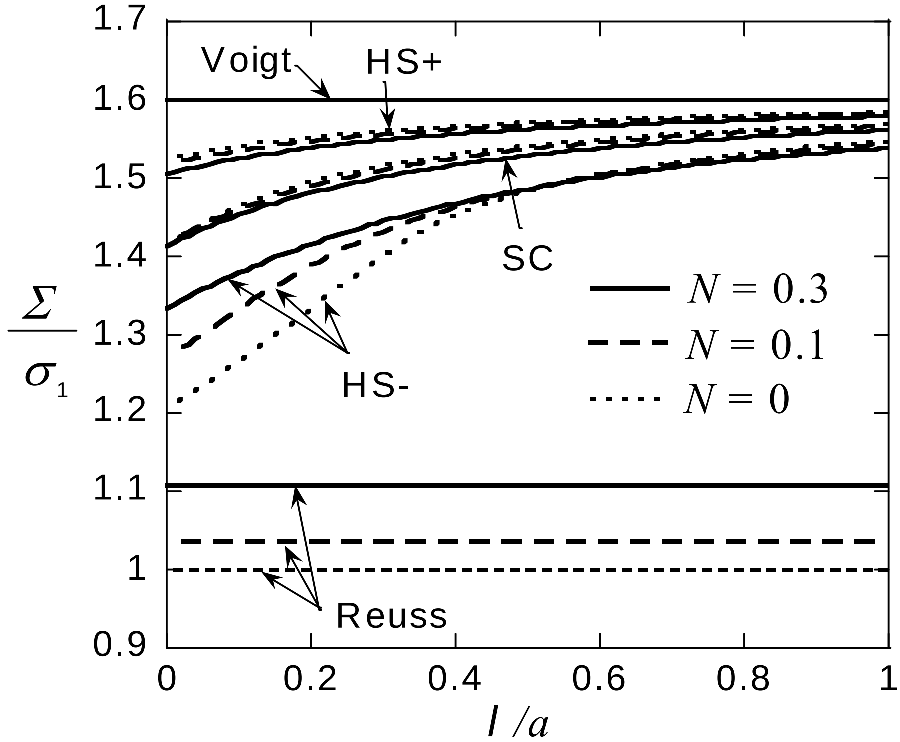


Figure 3

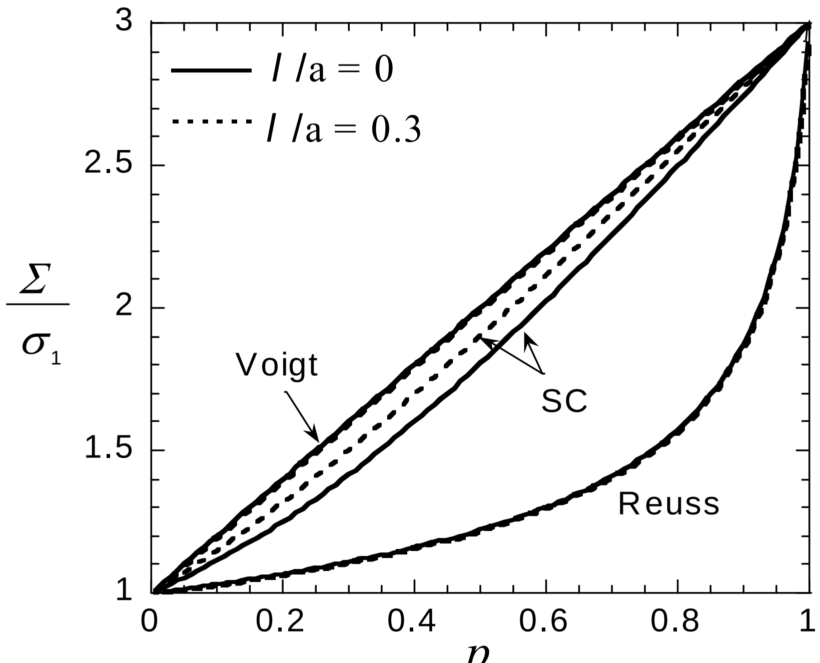


Figure 4