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A mathematical basis for strain-gradient plasticity theory—Part I: Scalar plastic multiplier

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ABSTRACT

Strain-gradient plasticity theories are reviewed in which some measure of the plastic strain rate is treated as an independent kinematic variable. Dislocation arguments are invoked in order to provide a physical basis for the hardening at interfaces. A phenomenological, flow theory version of gradient plasticity is constructed in which stress measures, work-conjugate to plastic strain and its gradient, satisfy a yield condition. Plastic work is also done at internal interfaces and a yield surface is postulated for the work-conjugate stress quantities at the interface. Thereby, the theory has the potential to account for grain size effects in polycrystals. Both the bulk and interfacial stresses are taken to be dissipative in nature and due attention is paid to ensure that positive plastic work is done. It is shown that the mathematical structure of the elasto-plastic strain-gradient theory has similarities to conventional rigid-plasticity theory. Uniqueness and extremum principles are constructed for the solution of boundary value problems.

1. Introduction

A number of gradient theories of plasticity have emerged over the past decade, and in each case some measure of the plastic strain rate and its spatial gradient enter a statement of the principle of virtual work, see for example Gudmundson (2004), Fleck and Hutchinson (2001), Gurtin (2000) and Gurtin and Anand (2008). Higher order boundary conditions naturally arise via the work principle, and in this sense the theories are well-posed. However, there is choice in the ‘best formulation’ as motivated by the underlying physics. The purpose of the present paper is to provide a mathematical basis for a revised flow theory version of the Fleck–Hutchinson theory, to extend this theory to include interface terms and to present the accompanying uniqueness and extremum principles. The inclusion of interface terms is similar in concept to that outlined by Gudmundson (2004) and Aifantis and Willis (2005, 2006), but differs in its implementation through concentrating on flow theory and plastic dissipation at interfaces. Recently, Fredriksson and Gudmundson (2005, 2007) have used a visco-plastic formulation to explore the relative role of dissipation at an interface between a strain-gradient plastic solid and an elastic solid. They considered the strengthening of a thin film on an elastic substrate and demonstrated that the yield strength of the film is increased by length-scale effects both in the bulk material and at the interface.

The subject of strain-gradient plasticity has been actively researched in recent years, and has been the topic of several dedicated colloquia, see for example volume 15, issue 1, of *Modelling and Simulation in Materials Science and Engineering*, January 2007. (The cited paper of Fredriksson and Gudmundson (2007) is in this issue.) The early versions of

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strain-gradient plasticity theory (e.g. Aifantis, 1984, 1987; Fleck and Hutchinson, 1997) assume that the material length scale is constant and does not evolve with plastic deformation. More recent theories (such as Gao et al., 1999) assume that the material length scale decreases with increasing plastic deformation. Detailed comparison with experimental data is required in order to assess the relative accuracy of these formulations for a range of problems, such as wire torsion, beam bending and indentation. There remains a need to show that the constitutive relations adopted for strain-gradient plasticity generate well-posed problems with a unique solution. The current work contributes towards this by providing uniqueness and extremum principles for the theory that is proposed.

1.1. Dissipative versus energetic stresses

The current theories of strain-gradient plasticity are motivated by the notion that geometrically necessary dislocations lead to enhanced strengthening (Nye, 1953; Ashby, 1970; Fleck et al., 1994). The degree to which the additional strengthening is mainly energetic or dissipative in nature remains an open issue. Gurtin (2000) argues that the density of geometrically necessary dislocations, as quantified by the Nye tensor (Nye, 1953) leads to an increase in free energy of the solid. However, direct experimental measurement demonstrates that the core energy of dislocations stored during plastic deformation is much smaller than the plastic work dissipated in dislocation motion. Consequently, one would expect that statistically stored and geometrically necessary dislocations would contribute more to plastic dissipation than to a change in free energy. The current work develops a simple theory which admits plastic dissipation but disregards changes in the free energy due to plastic deformation. Subsequent work (paper II) will address the more general case.

Internal interfaces such as grain boundaries act as additional barriers to slip. It is now argued that a major *local* effect of such interfaces is to lead to dissipation rather than to a change in internal energy. The long-range elastic stress fields associated with constrained plastic flow of a dislocation pile-up against a boundary are already accounted for in an elastic–plastic calculation: for example, elastic back-stresses at a dislocation pile-up near a grain boundary are present in a crystal plasticity calculation when a soft, yielded grain is embedded within stronger grains. To fix our ideas, consider the flow resistance associated with two extreme views of dislocation arrangement at a grain boundary, as sketched in Fig. 1: (a) single slip within one grain and no slip within the adjacent grain, and (b) single slip in one grain, and double slip within the adjacent grain. Consider each in turn:

(i) Allow edge dislocations on a set of parallel slip planes to be driven to the grain boundary under an applied shear stress (see Fig. 1a). The core energy of the dislocations will alter when the dislocations enter the grain boundary or its immediate vicinity, as the local environment is different at the boundary than in the bulk, but this will lead to a negligible change in the free energy of the solid. The presence of the dislocations at the boundary will also increase the local, dissipative barrier and lead to macroscopic strengthening. Additionally, long-range elastic back-stresses may or may not be induced, depending upon the surrounding elastic constraint by adjacent grains. We note that there is a jump in the plastic strain at the interface, and a finite surface Nye tensor exists at the interface, as defined by Gurtin and Needleman (2005); this surface tensor can be used to quantify the dislocation density at the boundary. The above discussion motivates the idea that the interface leads to macroscopic dissipative strengthening, with the level of strengthening characterised by the jump in plastic strain across the interface.

(ii) Assume instead that plastic slip occurs on both sides of the boundary, such that a finite average plastic strain exists at the interface, but with no jump in value (see Fig. 1b). For example, imagine that single slip occurs first within one grain, and then triggers double slip within the neighbouring grain. The dislocations in the second grain are nucleated at the boundary or from sources within the bulk of the second grain and are then attracted back to the boundary. An arrangement of dislocations is left on the boundary as sketched in the figure, but the surface Nye tensor vanishes. Despite this, the state of disorder at the boundary has changed. There is a small change in the free energy of the solid but the more major effect is an increase in flow resistance: subsequent dislocations encountering the grain boundary must undergo local interactions with these dislocations and bypass them. This is analogous to the hardening in the bulk when a dislocation must bypass dipoles left behind by previous dislocation interactions. We conclude that an increment in mean plastic strain at an interface induces dissipative strengthening, with a negligible increase in free energy.

The above considerations motivate the present study: we seek to establish a phenomenological strain-gradient theory which is purely dissipative in nature, and include the additional resistance to plastic flow from internal interfaces. We shall consider a phenomenological theory in which the plastic multiplier $\dot{\varepsilon}^{\mathrm{P}}$ (equal to the von Mises equivalent plastic strain rate) is treated as a free kinematic scalar; this multiplier and its spatial gradient are used directly in a principle of virtual work. Additionally, we shall include the contribution to internal work at internal interfaces Γ such as grain boundaries. The idea of treating some measure of the plastic strain rate as a free kinematic variable originated with Gurtin (2000). In theories of this type, the spatial gradient of plastic strain rate follows naturally, but there is no work term involving the gradient of elastic strain.

In contrast, the Fleck-Hutchinson (1997) theory treats the displacement $\boldsymbol{u}$ as the fundamental kinematic unknown, with the total strain and total strain gradient derived directly from it. In their flow theory version, the total strain and the total

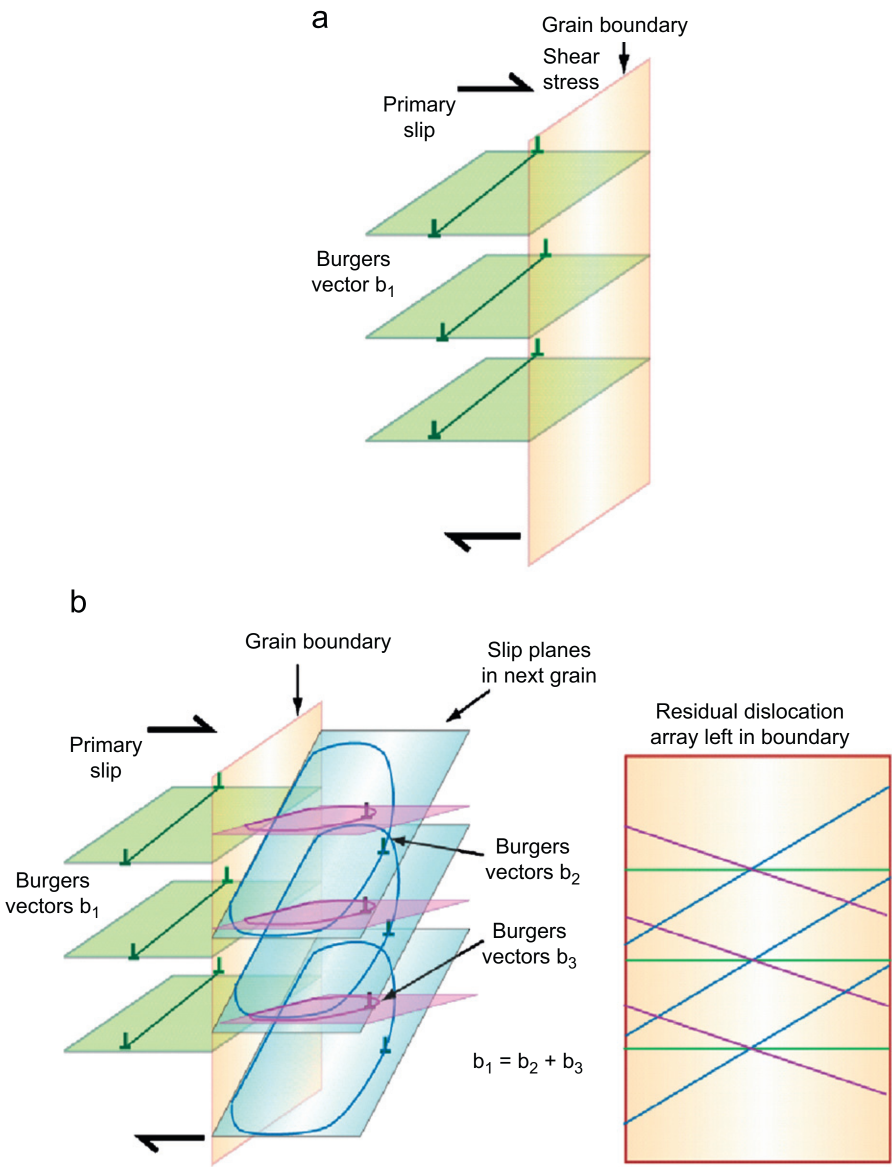

Fig. 1. Idealised dislocation reactions at a grain boundary. The grain on the left undergoes single slip; the grain on the right undergoes (a) no slip and (b) double slip.

strain gradient are decomposed additively into elastic and plastic components, with the elastic components specified in terms of the Cauchy stress and a higher order stress (a 3rd order tensor) via a generalized elasticity relation. The stress and higher order stress satisfy a generalized yield condition, and at yield the plastic rates of strain and strain gradient satisfy associated flow. In this manner, the theory is a natural extension of J_2 flow theory for a conventional elastic-plastic solid. Although the framework rests on firm mathematical and thermodynamic foundations, it assumes the existence of higher order elastic stresses and elastic strain gradients which are difficult to justify on physical grounds. Also, the theory has the curious feature that the plastic part of the strain gradient is not equal to the gradient of plastic strain: only the total strain and its gradient are related in a direct kinematic manner.

1.2. The internal plastic work

The present formulation assumes that plastic straining, and the spatial gradient of plastic straining, give rise to plastic dissipation. The changes in internal energy due to plastic strain and its gradient are neglected, and so the theory neglects energetic stresses conjugate to these variables. We leave these embellishments to a subsequent publication (paper II), as they necessitate the use of the plastic strain rate $\dot{\varepsilon}_{ij}^{\mathrm{PL}}$ as a free kinematic quantity in order to obtain classical kinematic hardening theory in the absence of strain gradients.

The starting point of the present work is the flow theory version of the Fleck and Hutchinson (2001) theory. In this theory the internal plastic work increment is written as

$$\int_V \{Q\delta\varepsilon^{\mathrm{P}} + \tau_i \delta\varepsilon^{\mathrm{P}}_{,i}\}\mathrm{d}V$$

thereby defining the work-conjugate stress quantities (Q, τ_i). We note in passing that Fleck and Hutchinson (2001) do not partition the above work statement into energetic and dissipative terms, and so positive plastic dissipation is not invoked. They give a prescription (Eqs. (24) and (25) of their paper) for the *rates* ($\dot{Q}, \dot{\tau}_i$):

$$\dot{Q} = h\dot{\varepsilon}^{\mathrm{P}} \quad \text{and} \quad \dot{\tau}_i = \ell_*^2 h \dot{\varepsilon}^{\mathrm{P}}_{,i} \tag{1.1}$$

in terms of a hardening modulus h and a single material length ℓ_*. (Fleck and Hutchinson (2001) also give a more sophisticated version involving three material length scales, but similar in mathematical structure to that of Eq. (1.1).) Their prescription ensures that $Q\delta\varepsilon^{\mathrm{P}} \geqslant 0$ but does not enforce that $\tau_i \delta\varepsilon^{\mathrm{P}}_{,i} \geqslant 0$, as already discussed by Gudmundson (2004) and Gurtin and Anand (2008).

In the current treatment we assume from the outset that (Q, τ_i) are dissipative quantities and we modify the Fleck-Hutchinson (2001) theory to a version that guarantees positive plastic work, $Q\delta\varepsilon^{\mathrm{P}}+\tau_i\delta\varepsilon^{\mathrm{P}}_{,i} \geqslant 0$. It is instructive to make use of a thermodynamic framework along the lines of Gurtin and Anand (2008), but using the original Fleck-Hutchinson (2001) notation.

2. The thermodynamic framework

We begin by postulating an internal work statement in terms of the increment in elastic strain, $\delta\varepsilon^{\mathrm{EL}}_{ij}$ and the increment in the plastic strain multiplier $\delta\varepsilon^{\mathrm{P}}$ along with its spatial gradient $\delta\varepsilon^{\mathrm{P}}_{,i}$. Internal work is also expended at internal interfaces Γ such as grain boundaries. Assume that the displacement rate $\dot{u}_i$ is continuous across these interfaces but $\dot{\varepsilon}^{\mathrm{P}}$ can jump in value across the interface. It is instructive to re-write $\dot{\varepsilon}^{\mathrm{P}}$ on each side of the interface in terms of a jump in value and the mean value, as follows. Write $[\dot{\varepsilon}^{\mathrm{P}}]$ as the jump $\dot{\varepsilon}^{\mathrm{P}}_+ - \dot{\varepsilon}^{\mathrm{P}}_-$ across a point of Γ, where the normal n_i to the interface points in the direction from the −ve side to the +ve side. Likewise, $\langle\dot{\varepsilon}^{\mathrm{P}}\rangle$ denotes the average value $(\dot{\varepsilon}^{\mathrm{P}}_+ + \dot{\varepsilon}^{\mathrm{P}}_-)/2$. Then

$$\dot{\varepsilon}^{\mathrm{P}}_+ = \tfrac{1}{2}[\dot{\varepsilon}^{\mathrm{P}}] + \langle\dot{\varepsilon}^{\mathrm{P}}\rangle \quad \text{and} \quad \dot{\varepsilon}^{\mathrm{P}}_- = -\tfrac{1}{2}[\dot{\varepsilon}^{\mathrm{P}}] + \langle\dot{\varepsilon}^{\mathrm{P}}\rangle \tag{2.1}$$

Define the internal virtual work increment δW_{int} as

$$\delta W_{\mathrm{int}} = \int_V \{\sigma_{ij}\delta\varepsilon^{\mathrm{EL}}_{ij} + Q\delta\varepsilon^{\mathrm{P}} + \tau_i\delta\varepsilon^{\mathrm{P}}_{,i}\}\mathrm{d}V + \int_\Gamma \{p[\delta\varepsilon^{\mathrm{P}}] + q\langle\delta\varepsilon^{\mathrm{P}}\rangle\}\mathrm{d}\Gamma \tag{2.2}$$

thereby introducing the work-conjugate quantities of the symmetric Cauchy stress σ_{ij} and (Q, τ_i) within the bulk, along with the work-conjugates (p, q) on the internal interfaces. The external virtual work δW_{ext} is

$$\delta W_{\mathrm{ext}} = \int_S \{T_i\delta u_i + t\delta\varepsilon^{\mathrm{P}}\}\mathrm{d}S \tag{2.3}$$

such that the external traction T_i is work-conjugate to the displacement increment, and t is work-conjugate to the scalar plastic strain increment. Then, equate the internal work to the external work in order to obtain the principle of virtual work:

$$\int_V \{\sigma_{ij}\delta\varepsilon^{\mathrm{EL}}_{ij} + Q\delta\varepsilon^{\mathrm{P}} + \tau_i\delta\varepsilon^{\mathrm{P}}_{,i}\}\mathrm{d}V + \int_\Gamma \{p[\delta\varepsilon^{\mathrm{P}}] + q\langle\delta\varepsilon^{\mathrm{P}}\rangle\}\mathrm{d}\Gamma = \int_S \{T_i\delta u_i + t\delta\varepsilon^{\mathrm{P}}\}\mathrm{d}S \tag{2.4}$$

It remains to obtain the governing equilibrium relations from Eq. (2.4). Following Fleck and Hutchinson (2001), the plastic strain increment $\delta\varepsilon^{\mathrm{PL}}_{ij}$ is taken to be co-directional with the deviatoric Cauchy stress s_{ij} but to scale in magnitude with $\delta\varepsilon^{\mathrm{P}}$, such that

$$\delta\varepsilon^{\mathrm{PL}}_{ij} = m_{ij}\delta\varepsilon^{\mathrm{P}} \quad \text{where} \quad m_{ij} \equiv \frac{3}{2}\frac{s_{ij}}{\sigma_{\mathrm{e}}} \tag{2.5}$$

Here, $\sigma_{\mathrm{e}} \equiv \sqrt{3s_{ij}s_{ij}/2}$ is the usual von Mises effective stress. The total strain increment $\delta\varepsilon_{ij}$ is given by the sum of the elastic and plastic contributions:

$$\delta\varepsilon_{ij} = \delta\varepsilon^{\mathrm{EL}}_{ij} + \delta\varepsilon^{\mathrm{PL}}_{ij} \tag{2.6}$$

Now integrate Eq. (2.2) by parts to obtain

$$\begin{aligned}\delta W_{\mathrm{int}} = &\int_V \{-\sigma_{ij,j}\delta u_i + (Q - \tau_{i,i} - \sigma_{\mathrm{e}})\delta\varepsilon^{\mathrm{P}}\}\mathrm{d}V \\ &+ \int_\Gamma \{p[\delta\varepsilon^{\mathrm{P}}] + q\langle\delta\varepsilon^{\mathrm{P}}\rangle - n_i[\tau_i\delta\varepsilon^{\mathrm{P}}]\}\mathrm{d}\Gamma + \int_S \{\sigma_{ij}n_j\delta u_i + \tau_i n_i\delta\varepsilon^{\mathrm{P}}\}\mathrm{d}S\end{aligned} \tag{2.7}$$

where S is the external surface of the solid, with unit outward normal n_i. Also, recall that we have adopted the convention that the normal n_i to the interface Γ points in the direction from the −ve side to the +ve side. It follows from the principle of virtual work (2.4), and from the identity $[fg] = \langle f \rangle [g] + [f] \langle g \rangle$, that

$$\text{in } V: \quad \sigma_{ij,j} = 0 \text{ and } Q - \tau_{i,i} - \sigma_e = 0 \tag{2.8}$$

$$\text{on } \Gamma: \quad p = n_i \langle \tau_i \rangle \text{ and } q = n_i [\tau_i] \tag{2.9}$$

$$\text{and on } S: \quad T_i = \sigma_{ij} n_j \text{ and } t = \tau_i n_i \tag{2.10}$$

These are the fundamental balance laws of equilibrium, valid everywhere, in the elastic and plastic zones.

Now introduce the internal energy $U = U(\varepsilon_{ij}{}^{\mathrm{EL}})$ and thereby define the energetic stress:

$$\sigma_{ij}^{\mathrm{EL}} = \frac{\partial U}{\partial \varepsilon_{ij}^{\mathrm{EL}}} \tag{2.11}$$

By the 2nd law, the change in internal energy is less than or equal to the internal work:

$$\int_V \delta U \, \mathrm{d}V \leqslant \delta W_{\mathrm{int}} \tag{2.12}$$

and this inequality is valid for any sub-volume of the solid, implying that

$$(\sigma_{ij} - \sigma_{ij}^{\mathrm{EL}}) \delta \varepsilon_{ij}^{\mathrm{EL}} + Q \delta \varepsilon^{\mathrm{P}} + \tau_i \delta \varepsilon_{,i}^{\mathrm{P}} \geqslant 0 \tag{2.13}$$

Now $\delta \varepsilon_{ij}^{\mathrm{EL}}$ can be of either sign and so $\sigma_{ij} \equiv \sigma_{ij}^{\mathrm{EL}}$. Then, we have the requirement of positive plastic work at each point within V:

$$Q \delta \varepsilon^{\mathrm{P}} + \tau_i \delta \varepsilon_{,i}^{P} \geqslant 0 \tag{2.14}$$

By a similar argument, the plastic dissipation at internal interfaces Γ is positive, and so Eq. (2.4) implies that

$$p[\delta \varepsilon^{\mathrm{P}}] + q \langle \delta \varepsilon^{\mathrm{P}} \rangle \geqslant 0 \tag{2.15}$$

at each point on Γ.

This is as far as thermodynamics takes us. We proceed to develop constitutive laws which relate (Q, τ_i) to $(\dot{\varepsilon}^{\mathrm{P}}, \dot{\varepsilon}_{,i}^{\mathrm{P}})$ within V, and (p, q) to $([\dot{\varepsilon}^{\mathrm{P}}], \langle \dot{\varepsilon}^{\mathrm{P}} \rangle)$ on Γ. This is achieved by introducing convex dissipation potentials which relate the plastic strain rate quantities to their work-conjugate generalized stresses. Positive plastic work is thereby assured and the requirements (2.14) and (2.15) are satisfied. With the objective of constructing a theory simple enough for use in applications, we assume that the dissipation potentials depend upon effective strain rates derived from quadratic forms within V and on Γ. The next step is to stipulate these effective strain rates.

3. The effective strain rates

Consider first the plastic dissipation within V. We follow Fleck and Hutchinson (2001) and make use of a generalized effective plastic strain rate $\dot{E}^{\mathrm{P}}$. Two definitions are adopted: a *one length-scale version* along the lines of the Aifantis theory, such that

$$(\dot{E}^{\mathrm{P}})^2 = (\dot{\varepsilon}^{\mathrm{P}})^2 + \ell_*^2 \dot{\varepsilon}_{,i}^{\mathrm{P}} \dot{\varepsilon}_{,i}^{\mathrm{P}} \tag{3.1}$$

and a *three length-scale version* involving the scalar invariants of the plastic strain rate $\dot{\varepsilon}_{ij}^{\mathrm{PL}}$ and of its spatial gradient $\dot{\varepsilon}_{ij,k}^{\mathrm{PL}}$. For this purpose, write the plastic strain-gradient rate as

$$\rho_{ijk} = \rho_{jik} \equiv \dot{\varepsilon}_{ij,k}^{\mathrm{PL}} \tag{3.2}$$

and recall from Fleck and Hutchinson (2001) that ρ_{ijk} can be decomposed uniquely into three mutually orthogonal tensors,

$$\rho_{ijk} = \rho_{ijk}^{(1)} + \rho_{ijk}^{(2)} + \rho_{ijk}^{(3)} \tag{3.3}$$

such that $\rho_{ijk}^{(m)} \rho_{ijk}^{(n)} = 0$ for $m \neq n$. Explicit definitions of $\rho_{ijk}^{(m)}$ for $m = 1, 2$ or 3 are given in terms of ρ_{ijk} by the relations (4) of Fleck and Hutchinson (2001). Now, ρ_{ijk} has only three independent invariants that are homogeneous of degree two, and so $\rho_{ijk}^{(m)} \rho_{ijk}^{(m)}$ for $m = 1, 2$ or 3 can be taken for that purpose. Thus, the three parameter version of the effective plastic strain rate $\dot{E}^{\mathrm{P}}$ can be written as

$$(\dot{E}^{\mathrm{P}})^2 = (\dot{\varepsilon}^{\mathrm{P}})^2 + \ell_1^2 \rho_{ijk}^{(1)} \rho_{ijk}^{(1)} + 4 \ell_2^2 \rho_{ijk}^{(2)} \rho_{ijk}^{(2)} + \frac{8}{3} \ell_3^2 \rho_{ijk}^{(3)} \rho_{ijk}^{(3)} \tag{3.4}$$

where again $(\dot{\varepsilon}^{\mathrm{P}})^2 = 2 \dot{\varepsilon}_{ij}^{\mathrm{PL}} \dot{\varepsilon}_{ij}^{\mathrm{PL}} / 3$ and the three material length parameters ℓ_i are required for dimensional consistency. The numerical factors in Eq. (3.4) ensure that the length scales are identical to those introduced originally by Fleck and Hutchinson (1997).

Note that the one parameter version (3.1) of $\dot{E}^{\mathrm{P}}$ is already expressed in terms of $\dot{\varepsilon}^{\mathrm{P}}$ and of its gradient $\dot{\varepsilon}_{,i}^{\mathrm{P}}$, as appearing in the principle of virtual work (2.4). Likewise, the three parameter version (3.4) can be re-written in terms of $\dot{\varepsilon}^{\mathrm{P}}$ and $\dot{\varepsilon}_{,i}^{\mathrm{P}}$ upon noting that

$$\rho_{ijk} = \dot{\varepsilon}^{\mathrm{PL}}_{ij,k} = m_{ij}\dot{\varepsilon}^{\mathrm{P}}_{,k} + m_{ij,k}\dot{\varepsilon}^{\mathrm{P}} \tag{3.5}$$

Perform the orthogonal decomposition (3.3) of ρ_{ijk} into its three constituent parts $\rho^{(m)}_{ijk}$; then, the expression (3.4) for the effective plastic strain rate $\dot{E}^{\mathrm{P}}$ can be re-cast as

$$(\dot{E}^{\mathrm{P}})^2 = (\dot{\varepsilon}^{\mathrm{P}})^2 + \bar{A}_{ij}\dot{\varepsilon}^{\mathrm{P}}_{,i}\dot{\varepsilon}^{\mathrm{P}}_{,j} + \bar{B}_i\dot{\varepsilon}^{\mathrm{P}}_{,i}\dot{\varepsilon}^{\mathrm{P}} + \bar{C}(\dot{\varepsilon}^{\mathrm{P}})^2 \tag{3.6}$$

The coefficients $(\bar{A}_{ij}, \bar{B}_i, \bar{C})$ depend upon the three material length scales ℓ_m and upon m_{ij}, and explicit expressions for them are given in Appendix A.1 of Fleck and Hutchinson (2001). (Note that over-bars are introduced in the present study to simplify later notation.) The one length-scale version (3.1) is not a special case of the three length-scale version, but formally Eq. (3.1) is obtained by specifying that $\bar{A}_{ij} = \ell_*^2\delta_{ij}$ and $\bar{B}_i = \bar{C} = 0$. For later convenience, we shall simplify the above notation of Fleck and Hutchinson (2001) by introducing the four-dimensional vector $\dot{\boldsymbol{e}} = (\dot{e}_{\mathrm{I}}) = (\dot{\varepsilon}^{\mathrm{P}}, \dot{\varepsilon}^{\mathrm{P}}_{,1}, \dot{\varepsilon}^{\mathrm{P}}_{,2}, \dot{\varepsilon}^{\mathrm{P}}_{,3})$. Then, Eqs. (3.1) and (3.6) can be re-cast in the form

$$(\dot{E}^{\mathrm{P}})^2 = A_{IJ}\dot{e}_I\dot{e}_J, \quad (I,J) \text{ sum over } 0,\ldots 3 \tag{3.7}$$

where the coefficients in the symmetric, positive definite 4×4 matrix (A_{IJ}) are related directly to $(\bar{A}_{ij}, \bar{B}_i, \bar{C})$ according to $A_{00} = 1 + \bar{C}$, $A_{0I} = A_{I0} = \frac{1}{2}\bar{B}_I$ for $I \in \{1,2,3\}$ and $A_{IJ} = \bar{A}_{IJ}$ for $I,J \in \{1,2,3\}$.

Second, consider the plastic dissipation at internal interfaces Γ, and recall that it scales with $([\dot{\varepsilon}^{\mathrm{P}}], \langle\dot{\varepsilon}^{\mathrm{P}}\rangle)$. We introduce an effective strain rate $\dot{D}$ and define it as the square root of the general quadratic form of $([\dot{\varepsilon}^{\mathrm{P}}], \langle\dot{\varepsilon}^{\mathrm{P}}\rangle)$. To do so, write $\dot{\boldsymbol{d}} = (\dot{d}_\alpha) = ([\dot{\varepsilon}^{\mathrm{P}}], \langle\dot{\varepsilon}^{\mathrm{P}}\rangle)$ as a two-dimensional vector. Next, construct

$$\dot{D}^2 = a_{\alpha\beta}\dot{d}_\alpha\dot{d}_\beta \tag{3.8}$$

with the repeated Greek suffixes denoting summation over 1 to 2; the 2×2 symmetric matrix $(a_{\alpha\beta})$ is positive definite.

In the following, we shall assume that the plastic dissipation depends directly upon $\dot{E}_{\mathrm{P}}$ within V and upon $\dot{D}$ on Γ. A visco-plastic formulation is now given, followed by the rate-independent case.

4. Visco-plastic version

We first introduce a power law creep potential, and then take the perfectly plastic limit. This ensures that positive plastic work is done.

4.1. Creep potential within V

First, we simplify our notation further by introducing the four-dimensional vector $\boldsymbol{r} = (r_I) = (Q, \tau_1, \tau_2, \tau_3)$. Note that r_I is work-conjugate to $\dot{e}_I$ within V. Then, we can write a creep potential $\phi(\dot{E}^{\mathrm{P}})$ within V as

$$\phi = \int r_I \mathrm{d}\dot{e}_I \equiv \frac{\sigma_0\dot{\varepsilon}_0}{N+1}\left(\frac{\dot{E}^{\mathrm{P}}}{\dot{\varepsilon}_0}\right)^{N+1} \tag{4.1}$$

where $(\sigma_0, \dot{\varepsilon}_0, N)$ are material constants in the usual notation for power law creep and $\dot{E}^{\mathrm{P}}$ has already been specified by Eq. (3.7). Introduce the work-conjugate Σ of $\dot{E}^{\mathrm{P}}$, such that

$$\Sigma = \frac{\partial\phi}{\partial\dot{E}^{\mathrm{P}}} = \sigma_0\left(\frac{\dot{E}^{\mathrm{P}}}{\dot{\varepsilon}_0}\right)^N \tag{4.2}$$

Then

$$r_I = \frac{\partial\phi}{\partial\dot{e}_I} = \frac{\Sigma}{\dot{E}^{\mathrm{P}}}A_{IJ}\dot{e}_J \tag{4.3}$$

and substitution into Eq. (3.7) provides the direct relation between Σ and r_I as

$$\Sigma^2 = D_{IJ}r_Ir_J \tag{4.4}$$

where $D_{IJ} \equiv (A^{-1})_{IJ}$. Note that the plastic work-rate reads

$$r_I\dot{e}_I = \Sigma\dot{E}^P = \sigma_0\dot{\varepsilon}_0\left(\frac{\dot{E}^{\mathrm{P}}}{\dot{\varepsilon}_0}\right)^{N+1} \geqslant 0 \tag{4.5}$$

as required by Eq. (2.14). The constitutive dual $\phi^*(\Sigma)$ of $\phi(\dot{E}^{\mathrm{P}})$ follows immediately as

$$\phi^*(\Sigma) = \sup_{\dot{E}^{\mathrm{P}}}(\Sigma\dot{E}^{\mathrm{P}} - \phi(\dot{E}^{\mathrm{P}})) = \frac{N}{N+1}\sigma_0\dot{\varepsilon}_0\left(\frac{\Sigma}{\sigma_0}\right)^{N+1/N} \tag{4.6}$$

upon making use of the choice (4.1) for $\phi(\dot{E}^{\mathrm{P}})$. The generalized strain rate $\dot{e}_I$ is related to r_I according to

$$\dot{e}_I = \frac{\partial \phi^*}{\partial r_I} = \frac{\dot{E}^{\mathrm{P}}}{\Sigma} D_{IJ} r_J \tag{4.7a}$$

This leads directly to the result

$$(\ln \dot{\varepsilon}^{\mathrm{P}})_{,I} = \frac{D_{IJ} r_J}{D_{0K} r_K} \quad \text{for } I = 1, 2 \text{ or } 3 \text{ and for } J, K \text{ summed over 0 to 3} \tag{4.7b}$$

4.2. Creep potential on Γ

In similar fashion, a creep potential $\omega(\dot{D})$ can be constructed on internal interfaces Γ. First, introduce the two-dimensional vector $\boldsymbol{s} = (s_\alpha) = (p, q)$ as the work-conjugate to $\dot{\boldsymbol{d}}$. Then, the assumed creep potential reads

$$\omega = \int \boldsymbol{s} \cdot d\dot{\boldsymbol{d}} \equiv \frac{\sigma'_0 \dot{\varepsilon}'_0}{M+1} \left(\frac{\dot{D}}{\dot{\varepsilon}'_0} \right)^{M+1} \tag{4.8}$$

where $(\sigma'_0, \dot{\varepsilon}'_0, M)$ are additional creep constants and $\dot{D}$ is given by Eq. (3.8). The work-conjugate to $\dot{D}$ is

$$S = \frac{\partial \omega}{\partial \dot{D}} = \sigma'_0 \left(\frac{\dot{D}}{\dot{\varepsilon}'_0} \right)^{M} \tag{4.9}$$

and likewise

$$s_\alpha = \frac{\partial \omega}{\partial \dot{d}_\alpha} = \frac{S}{\dot{D}} a_{\alpha\beta} \dot{d}_\beta \tag{4.10}$$

Substitution of Eq. (4.10) into Eq. (3.8) provides the direct relation between S and s_α as

$$S^2 = b_{\alpha\beta} s_\alpha s_\beta \tag{4.11}$$

where $b_{\alpha\beta} \equiv (a^{-1})_{\alpha\beta}$. The plastic work-rate at any point on Γ is

$$s_\alpha \dot{d}_\alpha = S\dot{D} = \sigma'_0 \dot{\varepsilon}'_0 \left(\frac{\dot{D}}{\dot{\varepsilon}'_0} \right)^{M+1} \geqslant 0 \tag{4.12}$$

and the constitutive dual $\omega^*(S)$ of $\omega(\dot{D})$ reads

$$\omega^*(S) = \sup_{\dot{D}} (S\dot{D} - \omega(\dot{D})) = \frac{M}{M+1} \sigma'_0 \dot{\varepsilon}'_0 \left(\frac{S}{\sigma'_0} \right)^{M+1/M} \tag{4.13}$$

The generalized strain rate $\dot{d}_\alpha$ on Γ is related to s_α according to

$$\dot{d}_\alpha = \frac{\partial \omega^*}{\partial s_\alpha} = \frac{\dot{D}}{S} b_{\alpha\beta} s_\beta \tag{4.14}$$

4.3. Minimum principle for the rate-dependent constitutive law

A minimum principle can be stated for the rate-dependent problem by following the general prescription of Suo (1997) and Cocks et al. (1999). First, define the free energy of the solid as

$$H(u_i, \varepsilon^{\mathrm{P}}) \equiv \int_V \{U(\varepsilon^{\mathrm{EL}}_{ij})\} \mathrm{d}V - \int_{S_T} \{T_i^0 u_i + t^0 \varepsilon^{\mathrm{P}}\} \mathrm{d}S \tag{4.15}$$

and note that the rate of free energy is

$$\dot{H}(\dot{u}_i, \dot{\varepsilon}^{\mathrm{P}}) = \int_V \{\sigma_{ij}(\dot{\varepsilon}_{ij} - \dot{\varepsilon}^{\mathrm{PL}}_{ij})\} \mathrm{d}V - \int_{S_T} \{T_i^0 \dot{u}_i + t^0 \dot{\varepsilon}^{\mathrm{P}}\} \mathrm{d}S \tag{4.16}$$

Second, write a dissipation potential as

$$\Psi(\dot{\varepsilon}^{\mathrm{P}}) \equiv \int_V \{\phi(\dot{E}^{\mathrm{P}})\} \mathrm{d}V + \int_\Gamma \{\omega(\dot{D})\} \mathrm{d}\Gamma \tag{4.17}$$

Then, construct a combined potential $\Omega = \Omega(\dot{u}_i, \dot{\varepsilon}^{\mathrm{P}}) \equiv \dot{H} + \Psi$. The first variation of Ω with respect to $(\dot{u}_i, \dot{\varepsilon}^{\mathrm{P}})$ delivers the principle of virtual work (2.4) along with the constitutive statements (4.3) and (4.10).

5. Rate-independent version

Now take the rate-independent limit, $N \to 0$ and $M \to 0$, and introduce strain hardening such that σ_0 is replaced by $\Sigma_Y(E^{\mathrm{P}})$ and σ'_0 is replaced by $S_Y(D)$, where $E^{\mathrm{P}} = \int \dot{E}^{\mathrm{P}}\,\mathrm{d}t$ and $D = \int \dot{D}\,\mathrm{d}t$. We shall show that the response closely resembles that of a rigid, hardening solid, endowed with a yield surface and a hardening law. We consider first the response within the bulk, and then consider the yield response at internal interfaces.

5.1. Yield surface and hardening rule in V

In the rate-independent limit the flow rule (4.7) reduces to

$$\dot{e}_I = \frac{\partial \phi^*}{\partial r_I} = \dot{E}^{\mathrm{P}} \frac{\partial \Sigma}{\partial r_I}, \quad \Sigma(\boldsymbol{r}) = \Sigma_Y(E^{\mathrm{P}}) \tag{5.1}$$

provided $\dot{E}^{\mathrm{P}} > 0$. We can consider Eq. (5.1) to be a statement of normality in the four-dimensional stress space of $\boldsymbol{r}$. The plastic strain rate vector $\dot{\boldsymbol{e}}$ is aligned with the outward normal to the yield surface $\Sigma(\boldsymbol{r}) = \Sigma_Y(E^{\mathrm{P}})$. When $\Sigma(\boldsymbol{r})$ is less than $\Sigma_Y(E^{\mathrm{P}})$ the response is elastic and $\dot{E}^{\mathrm{P}} = \dot{e}_I = 0$. When $\dot{E}^{\mathrm{P}}$ is non-vanishing, $\boldsymbol{r}$ is given in terms of $\dot{\boldsymbol{e}}$ via Eq. (4.3), with $\Sigma = \Sigma_Y(E^{\mathrm{P}})$. This is analogous to the response of a rigid-hardening, conventional solid. When the solid is behaving in an elastic manner, $\boldsymbol{r}$ is indeterminate, but must still satisfy the balance law (2.8ii). Recall that in the rigid zone of a rigid-hardening solid, the stress is indeterminate but must still satisfy the equilibrium relation (2.8i).

But what about the hardening rule? At yield, we have $\Sigma = \Sigma_Y(E^{\mathrm{P}})$ and so plastic loading implies the consistency relation $\dot{\Sigma} = h(E^{\mathrm{P}})\dot{E}^{\mathrm{P}}$ where $h \equiv \mathrm{d}\Sigma_Y/\mathrm{d}E^{\mathrm{P}}$. Consequently, Eq. (5.1) can be written in the usual form for associated plastic flow:

$$\dot{\boldsymbol{e}} = \frac{1}{h}\frac{\partial \Sigma}{\partial \boldsymbol{r}}\left(\frac{\partial \Sigma}{\partial \boldsymbol{r}} \cdot \dot{\boldsymbol{r}}\right) \tag{5.2}$$

Since Σ is homogeneous and of degree one in $\boldsymbol{r}$ we have $\boldsymbol{r} \cdot \dot{\boldsymbol{e}} = (\boldsymbol{r} \cdot (\partial \Sigma/\partial \boldsymbol{r}))\dot{\Sigma}/h = \Sigma\dot{\Sigma}/h = \Sigma\dot{E}^{\mathrm{P}} \geqslant 0$ as demanded by the second law (2.14).

Given $\dot{\boldsymbol{e}}$, what is $\dot{\boldsymbol{r}}$? The component of $\dot{\boldsymbol{r}}$ along the normal to the yield surface is known from the constitutive law, *but not* its tangential component. But given $\dot{\boldsymbol{e}}$ and $\dot{\varepsilon}_{ij}$, the stress rate $\dot{\sigma}_{ij}$ is known, as follows. Assume the usual relation between stress rate and elastic strain rate:

$$\dot{\sigma}_{ij} = L_{ijkl}(\dot{\varepsilon}_{kl} - \dot{\varepsilon}^{\mathrm{PL}}_{kl}) \tag{5.3}$$

where L_{ijkl} is the elastic modulus. The plastic strain rate is specified in terms of σ_{ij} via $\dot{\varepsilon}^{\mathrm{PL}}_{ij} = m_{ij}\dot{\varepsilon}^{\mathrm{P}}$. Consequently, $\dot{\sigma}_{ij}$ is known in terms of $\dot{\varepsilon}_{ij}$ and $\dot{\varepsilon}^{\mathrm{P}}$.

5.2. Yield surface and hardening rule on Γ

In the rate-independent limit, we consider the possibility of yield being attained at internal interfaces. The flow rule (4.14) reduces to

$$\dot{d}_\alpha = \frac{\partial \omega^*}{\partial s_\alpha} = \dot{D}\frac{\partial S}{\partial s_\alpha}, \quad S(\boldsymbol{s}) = S_Y(D) \tag{5.4}$$

provided $\dot{D} > 0$. Eq. (5.4) is a statement of normality in the two-dimensional stress space of $\boldsymbol{s}$: the plastic strain rate vector $\dot{\boldsymbol{d}}$ is aligned with the outward normal to the yield surface $S(\boldsymbol{s}) = S_Y(D)$. When $S(\boldsymbol{s})$ is less than $S_Y(D)$ the response is elastic and $\dot{D} = \dot{d}_\alpha = 0$. Dually, for the case of non-vanishing $\dot{D}$, $\boldsymbol{s}$ is given in terms of $\dot{\boldsymbol{d}}$ via (4.10), with $S = S_Y(D)$. When the solid is behaving in an elastic manner, $\boldsymbol{s}$ is indeterminate, but must still satisfy the balance law (2.9).

Plastic loading implies the consistency relation $\dot{S} = g(D)\dot{D}$ where $g \equiv \mathrm{d}S_Y/\mathrm{d}D$. Consequently, Eq. (5.4) can be written in the usual form for associated plastic flow:

$$\dot{\boldsymbol{d}} = \frac{1}{g}\frac{\partial S}{\partial \boldsymbol{s}}\left(\frac{\partial S}{\partial \boldsymbol{s}} \cdot \dot{\boldsymbol{s}}\right) \tag{5.5}$$

5.3. Solution strategy for boundary value problems using the rate-independent formulation

Hereon, we shall focus on the rate-independent case. It is shown that a unique solution to a boundary problem requires knowledge of both the surface traction and its rate. The solution process involves the following steps, and each is detailed in subsequent sections.

(i) In the current state, the stress $\sigma_{ij}(\boldsymbol{x})$, displacement $u_i(\boldsymbol{x})$ and plastic strain $\varepsilon^{PL}_{ij}(\boldsymbol{x})$ are known everywhere. In general, the solid is partitioned into active plastic and elastic zones. We invoke a uniqueness principle I to argue that the generalized stress ($\boldsymbol{r}$, $\boldsymbol{s}$) is unique within each active plastic zone, while the plastic multipliers ($\dot{E}^{\mathrm{P}}, \dot{D}$) are known uniquely up to an arbitrary constant plastic multiplier $\dot{\Lambda}$. For the case where $\dot{\varepsilon}^{\mathrm{P}}$ is specified at some point, the associated multiplier $\dot{\Lambda}$ is determined.

(ii) We apply a minimum principle I over all kinematically admissible trial fields $\dot{\varepsilon}^{\mathrm{P}*}(\boldsymbol{x})$, with given traction boundary data, in order to obtain $(\boldsymbol{r}, \boldsymbol{s})$ uniquely in the active plastic zones. In the elastic zone(s), $(\boldsymbol{r}, \boldsymbol{s})$ are indeterminate. This principle will also deliver a unique distribution of $(\dot{E}^{\mathrm{P}}, \dot{D})$ but scaled in value by an arbitrary constant plastic multiplier $\dot{\Lambda}$. Consequently, $(\dot{\boldsymbol{e}}, \dot{\boldsymbol{d}})$ are made non-unique by the arbitrary nature of $\dot{\Lambda}$. This minimum principle delivers an upper bound restricting the traction t. A dual formulation can be invoked with a minimization conducted over all equilibrium trial fields $(\boldsymbol{r}^*, \boldsymbol{s}^*)$ in order to obtain a corresponding lower bound. These two minimum principles (and the associated uniqueness principle I) are close analogues to those derived by Hill (1951) for the stress state in the active plastic zone of a rigid-hardening solid.

(iii) With $\sigma_{ij}(\boldsymbol{x})$ known everywhere, and $(\boldsymbol{r}, \boldsymbol{s})$ but not $\dot{\Lambda}$ known in the active plastic zone, now invoke a uniqueness principle II to argue that $\dot{\Lambda}$ can be obtained uniquely in each active plastic zone, provided traction rate data are known on the boundary.

(iv) Finally, apply a minimum principle II involving the traction rate boundary data in order to obtain unique values for $\dot{u}_i$ (and $\dot{\Lambda}$ if necessary). The plastic strain rates $(\dot{\boldsymbol{e}}, \dot{\boldsymbol{d}})$ follow immediately once the plastic multipliers are known. This allows us to obtain the stress rate $\dot{\sigma}_{ij}$ and thereby update the stress state. The above procedure is now repeated. An alternative dual formulation can be used to obtain $(\dot{\sigma}_{ij}, \dot{\Sigma}, \dot{S})$ uniquely. These second minimum principles (and the associated uniqueness principle II) are analogues of the extremum and uniqueness principles devised by Hill (1956) for a conventional rigid-hardening solid.

6. Uniqueness principle for the stress quantities (*r*, *s*)

Assume that the stress $\sigma_{ij}(\boldsymbol{x})$ is known everywhere in the rate-independent elasto-plastic strain-gradient solid. Consider two solutions for $(\boldsymbol{r}, \boldsymbol{s})$ and for $(\dot{u}_i, \dot{\varepsilon}^{\mathrm{P}})$, and denote one solution by an asterisk and the other without an asterisk. The difference between the two solutions is denoted by the symbol Δ. Then, the principle of virtual work (2.4) gives

$$0 = \int_S \{\Delta T_i \Delta \dot{u}_i + \Delta t \Delta \dot{\varepsilon}^{\mathrm{P}}\} \mathrm{d}S = \int_V \{\Delta\sigma_{ij}\Delta\dot{\varepsilon}^{\mathrm{EL}}_{ij} + \Delta r_I \Delta \dot{e}_I\} \mathrm{d}V + \int_\Gamma \{\Delta s_\alpha \Delta \dot{d}_\alpha\} \mathrm{d}\Gamma \tag{6.1}$$

We assert that $\Delta\sigma_{ij} = 0$. Now Eqs. (5.1) and (5.4) state that $(\dot{\boldsymbol{e}}, \dot{\boldsymbol{d}})$ are normal to their respective yield surfaces. Also, the yield surfaces $\Sigma(\boldsymbol{r}) = \Sigma_Y$ and $S(\boldsymbol{s}) = S_Y$ are convex, and so

$$\text{in } V: \quad (r_I^* - r_I)\dot{e}_I^* \geqslant 0 \text{ and } (r_I - r_I^*)\dot{e}_I \geqslant 0 \tag{6.2}$$

$$\text{on } \Gamma: (s_\alpha^* - s_\alpha)\dot{d}_\alpha^* \geqslant 0 \text{ and } (s_\alpha - s_\alpha^*)\dot{d}_\alpha \geqslant 0 \tag{6.3}$$

Consequently, $\Delta r_I \Delta \dot{e}_I \geqslant 0$ unless $(r_I{}^* = r_I)$ and $\Delta s_\alpha \Delta \dot{d}_\alpha \geqslant 0$ unless $(s_\alpha{}^* = s_\alpha)$. Unique values for $(\boldsymbol{r}, \boldsymbol{s})$ dictate unique directions for $(\dot{\boldsymbol{e}}, \dot{\boldsymbol{d}})$ but the values of the plastic multipliers $(\dot{E}^{\mathrm{P}}, \dot{D})$ may be non-unique. Thus, $\Delta \dot{e}_I$ and $\Delta \dot{d}_\alpha$ may be non-zero. We conclude that $(\boldsymbol{r}, \boldsymbol{s})$ are unique in the active plastic zone, but the values of the plastic multipliers $(\dot{E}^{\mathrm{P}}, \dot{D})$ may be non-unique. This lack of uniqueness can be made precise as follows. Recall that (4.7b) holds for the rate-independent case as well as the rate-dependent case; this identity leads directly to the result that uniqueness of $\boldsymbol{r}$ implies uniqueness of $\dot{\varepsilon}^{\mathrm{P}}(\boldsymbol{x})$ up to an arbitrary multiplicative constant $\dot{\Lambda}$ within each active plastic zone. Further, the value of $\dot{\Lambda}$ is known for any active plastic zone when the value of $\dot{\varepsilon}^{\mathrm{P}}(\boldsymbol{x})$ is prescribed on a portion of the boundary of the plastic zone. This can occur when a portion S_u of the surface of the body is subjected to prescribed $\dot{\varepsilon}^{\mathrm{P}}(\boldsymbol{x})$.

7. Minimum principle I to obtain the stress quantities (*r*, *s*)

Assume that the stress $\sigma_{ij}(\boldsymbol{x})$ and the plastic strain ε_{ij}^{PL} are known everywhere. Write $(\dot{\boldsymbol{e}}, \dot{\boldsymbol{d}}; \boldsymbol{r}, \boldsymbol{s})$ as the actual solution, and recall that $(\boldsymbol{r}, \boldsymbol{s})$ is given by Eqs. (4.3) and (4.10) in the active plastic zone and is indeterminate in the elastic zone. Consider any kinematically admissible trial fields $(\dot{\boldsymbol{e}}^*, \dot{\boldsymbol{d}}^*)$, and note that $\boldsymbol{r}^*$ follows immediately from Eq. (4.3) provided $\dot{\boldsymbol{e}}^* \neq 0$. If $\dot{\boldsymbol{e}}^* = 0$ then $\boldsymbol{r}^*$ is indeterminate but is within or on the yield surface. Likewise, $\boldsymbol{s}^*$ follows immediately from Eq. (4.10) provided $\dot{\boldsymbol{d}}^* \neq 0$. If $\dot{\boldsymbol{d}}^* = 0$ then $\boldsymbol{s}^*$ is indeterminate but is within or on the yield surface. Then, the actual solution for $(\dot{\boldsymbol{e}}, \dot{\boldsymbol{d}}; \boldsymbol{r}, \boldsymbol{s})$ satisfies the following minimum statement:

$$H = \int_{S_U} \{t\dot{\varepsilon}^{P0}\} \mathrm{d}S = \inf_{\left(\dot{\varepsilon}^{P^*} \geq 0\right)} \int_V \{\Sigma_Y \dot{E}^{\mathrm{P}^*} - \sigma_e \dot{\varepsilon}^{\mathrm{P}^*}\} \mathrm{d}V + \int_\Gamma \{S_Y \dot{D}^*\} \mathrm{d}\Gamma - \int_{S_T} \{t^0 \dot{\varepsilon}^{\mathrm{P}^*}\} \mathrm{d}S \tag{7.1}$$

Proof. The 'maximum plastic work principle for a material element' (6.2i) and (6.3i), and the balance laws (2.8ii) and (2.9) imply that

$$\int_V \{r_I^* \dot{e}_I^*\} \mathrm{d}V + \int_\Gamma \{s_\alpha^* \dot{d}_\alpha^*\} \mathrm{d}\Gamma \geqslant \int_V \{r_I \dot{e}_I^*\} \mathrm{d}V + \int_\Gamma \{s_\alpha \dot{d}_\alpha^*\} \mathrm{d}\Gamma = \int_V \{\sigma_e \dot{\varepsilon}^{\mathrm{P}^*}\} \mathrm{d}V + \int_S \{\tau_i n_i \dot{\varepsilon}^{\mathrm{P}^*}\} \mathrm{d}S \tag{7.2}$$

Upon recalling that $r_I^* \dot{e}_I^* = \Sigma_Y \dot{E}^{P^*}$ and that $s_\alpha^* \dot{d}_\alpha^* = S_Y \dot{D}$ the above inequality can be re-arranged to the form of Eq. (7.1). This minimum principle delivers unique values for $(\boldsymbol{r}, \boldsymbol{s})$ in the active plastic zone. The plastic strain rate $\dot{\varepsilon}^{\mathrm{P}}(\boldsymbol{x})$ and the plastic multipliers $(\dot{E}^{\mathrm{P}}, \dot{D})$ are also unique up to the value of an arbitrary multiplicative constant $\dot{\Lambda}$, as discussed in Section 6. In order to define $\dot{\Lambda}$ unambiguously, introduce a strain rate field of 'unit magnitude' $\hat{\varepsilon}(\boldsymbol{x})$ within each active plastic zone V_{a}, such that

$$\frac{1}{V_{\mathrm{a}}} \int_V \hat{\varepsilon}(\boldsymbol{x}) \mathrm{d}V_{\mathrm{a}} = 1 \tag{7.3}$$

Then, $\dot{\Lambda}$ is the scaling factor for the plastic strain rate and we have

$$\dot{\varepsilon}^{\mathrm{P}}(\boldsymbol{x}) = \dot{\Lambda} \hat{\varepsilon}(\boldsymbol{x}) \tag{7.4}$$

We emphasise that Eq. (7.1) delivers both $\hat{\varepsilon}(\boldsymbol{x})$ and $\dot{\Lambda}$ when a non-zero $\dot{\varepsilon}^{\mathrm{P}}(\boldsymbol{x})$ is prescribed over a portion of the boundary of the solid; otherwise, Eq. (7.1) provides $\hat{\varepsilon}(\boldsymbol{x})$ *but not* $\dot{\Lambda}$. A finite element implementation of the minimum principle I is sketched in Appendix A.

The minimum principle (7.1) can be stated in an alternative form to provide additional insight. Write

$$H^* = \int_V \{\Sigma_Y \dot{E}^{P^*} - \sigma_e \dot{\varepsilon}^{P^*}\} \mathrm{d}V + \int_\Gamma \{S_Y \dot{D}^*\} \mathrm{d}\Gamma - \int_{S_T} \{t^0 \dot{\varepsilon}^{P^*}\} \mathrm{d}S \tag{7.5}$$

for any trial solution and

$$H = \int_V \{\Sigma_Y \dot{E}^{\mathrm{P}} - \sigma_e \dot{\varepsilon}^{\mathrm{P}}\} \mathrm{d}V + \int_\Gamma \{S_Y \dot{D}\} \mathrm{d}\Gamma - \int_{S_T} \{t^0 \dot{\varepsilon}^{\mathrm{P}}\} \mathrm{d}S \tag{7.6}$$

for the actual, equilibrium solution. Then, the difference $\Delta H \equiv H^* - H \geqslant 0$ can be written as

$$\Delta H = \int_V \{(Q^* - \tau_{i,i}^* - \sigma_e) \dot{\varepsilon}^{P^*}\} \mathrm{d}V + \int_{S_U} \{(\tau_i^* n_i - \tau_i n_i) \dot{\varepsilon}^{\mathrm{P}}\} \mathrm{d}S + \int_\Gamma \{(p^* - n_i \langle \tau_i^* \rangle)[\dot{\varepsilon}^{P^*}] + (q^* - n_i[\tau_i^*]) \langle \dot{\varepsilon}^{P^*} \rangle\} \, \mathrm{d}\Gamma \tag{7.7}$$

via the principle of virtual work (2.4) and the balance laws (2.8)–(2.10). The integrands in each of the integrals vanish when it is an equilibrium solution, as demanded by Eqs. (2.8)–(2.10). It follows immediately from (7.7) that the Euler equations from Eq. (7.1) are (2.8ii), (2.9) and (2.10ii).

7.1. Dual formulation

A dual formulation exists to Eq. (7.1). Instead of considering trial kinematic fields we now consider trial equilibrium fields $(\boldsymbol{r}^*, \boldsymbol{s}^*)$ that do not violate the yield condition at any point. Write the actual solution as $(\dot{\boldsymbol{e}}, \dot{\boldsymbol{d}}; \boldsymbol{r}, \boldsymbol{s})$. Then, we find that

$$\begin{aligned} I &= \int_{S_U} \{n_i \tau_i \dot{\varepsilon}^{\mathrm{P0}}\} \mathrm{d}S = \sup_{(\boldsymbol{r}^*, \boldsymbol{s}^*)} \int_{S_U} \{n_i \tau_i^* \dot{\varepsilon}^{\mathrm{P0}}\} \mathrm{d}S \\ &= \sup_{(\boldsymbol{r}^*, \boldsymbol{s}^*)} \int_V \{Q^* \dot{\varepsilon}^{\mathrm{P}} + \tau_i^* \dot{\varepsilon}_{,i}^{\mathrm{P}} - \sigma_e \dot{\varepsilon}^{\mathrm{P}}\} \, \mathrm{d}V + \int_\Gamma \{s_\alpha^* \dot{d}_\alpha\} \mathrm{d}\Gamma - \int_{S_T} \{n_i \tau_i^* \dot{\varepsilon}^{\mathrm{P}}\} \mathrm{d}S \end{aligned} \tag{7.8}$$

with equality attained when $(\boldsymbol{r}^*, \boldsymbol{s}^*)$ is the actual solution. This statement is analogous to Hill's (1948) maximum plastic work principle. The proof is a direct consequence of (6.2ii) and (6.3ii), along with the principle of virtual work (2.4). We conclude that (7.8) provides a lower bound restriction on t while (7.1) provides an upper bound, in direct correspondence to the classical theorems of limit load analysis.

8. Uniqueness principle for the plastic multipliers

Assume that the stress $\sigma_{ij}(\boldsymbol{x})$ is known everywhere, and $(\boldsymbol{r}, \boldsymbol{s})$ are known in the active plastic zone. We have already noted from minimum principle I and its associated uniqueness principle that $(\dot{E}^{\mathrm{P}}, \dot{D})$ within each active plastic zone is unique up to the value of a constant multiplier $\dot{\Lambda}$. We shall now show that a stipulation of the traction rate data is sufficient to ensure a unique solution for $\dot{\Lambda}$ and thereby $(\dot{E}^{\mathrm{P}}, \dot{D})$.

Consider two solutions for $(\dot{\boldsymbol{r}}, \dot{\boldsymbol{s}})$ and for $(\dot{u}_i, \dot{\varepsilon}^{\mathrm{P}})$, and denote the first solution by a superscript asterisk (*), and the second solution by the asterisk absent. The first solution minus the second solution is labelled by the symbol Δ. The plastic rates $(\dot{E}^{\mathrm{P}}, \dot{D})$ are linear in $\dot{\Lambda}$, while the rates $(\dot{E}^{P^*}, \dot{D}^*)$ are linear in $\dot{\Lambda}^*$. Then, the principle of virtual work (2.4) implies

$$0 = \int_S \{\Delta \dot{T}_i \Delta \dot{u}_i + \Delta \dot{t} \Delta \dot{\varepsilon}^{\mathrm{P}}\} \mathrm{d}S = \int_V \{\Delta \dot{\sigma}_{ij} \Delta \dot{\varepsilon}_{ij}^{\mathrm{EL}} + \Delta \dot{r}_I \Delta \dot{e}_I\} \mathrm{d}V + \int_\Gamma \{\Delta \dot{s}_\alpha \Delta \dot{d}_\alpha\} \mathrm{d}\Gamma \tag{8.1}$$

Consider the integrand on the right-hand side of (8.1). We shall show that it is positive unless the two solutions coincide. Assume that the elastic modulus L_{ijkl} is positive definite. Then

$$\Delta \dot{\sigma}_{ij} \Delta \dot{\varepsilon}_{ij}^{\mathrm{EL}} = L_{ijkl} \Delta \dot{\varepsilon}_{ij}^{\mathrm{EL}} \Delta \dot{\varepsilon}_{kl}^{\mathrm{EL}} \geqslant 0 \tag{8.2}$$

Second, consider $\Delta\dot{r}_I\Delta\dot{e}_I$. Recall that $\dot{\boldsymbol{e}} = \dot{E}^{\mathrm{P}}\partial\Sigma/\partial\boldsymbol{r}$ from Eq. (5.1) and so $\Delta\dot{\boldsymbol{e}} = (\Delta\dot{E}^{\mathrm{P}})\partial\Sigma/\partial\boldsymbol{r}$. Also, $(\Delta\dot{r}_I)\partial\Sigma/\partial r_I = \Delta\dot{\Sigma}$, and consequently

$$\Delta\dot{r}_I\Delta\dot{e}_I = \Delta\dot{\Sigma}\Delta\dot{E}^{\mathrm{P}} \tag{8.3}$$

A repeat of this argument for the interface terms gives immediately that

$$\Delta\dot{s}_\alpha\Delta\dot{d}_\alpha = \Delta\dot{S}\Delta\dot{D} \tag{8.4}$$

Now consider four distinct cases, as done by Hill (1956) for the conventional, rigid-hardening solid.

(i) $\dot{E}^{\mathrm{P}} > 0$, $\dot{E}^{\mathrm{P}^*} > 0$ (i.e. $\dot{\Lambda} > 0$, $\dot{\Lambda}^* > 0$). Then, $\Delta\dot{\Sigma}\Delta\dot{\Sigma}^{\mathrm{P}} = h(\Delta\dot{E}^{\mathrm{P}})^2 \geqslant 0$ provided $h > 0$.

(ii) $\dot{E}^{\mathrm{P}} > 0$, $\dot{E}^{\mathrm{P}^*} = 0$ (i.e. $\dot{\Lambda} > 0$, $\dot{\Lambda}^* = 0$). Then, $\dot{\Sigma}^*_Y = 0$ and $\dot{\Sigma}^* \leqslant 0$. Consequently, $\Delta\dot{E}^{\mathrm{P}} = \dot{E}^{\mathrm{P}^*} - \dot{E}^{\mathrm{P}} < 0$ and

$$\Delta\dot{\Sigma}\Delta\dot{\Sigma}^{\mathrm{P}} = (\dot{\Sigma}^* - \dot{\Sigma})\Delta\dot{E}^{\mathrm{P}} \geqslant -\dot{\Sigma}\Delta\dot{E}^{\mathrm{P}} = \dot{\Sigma}\dot{E}^{\mathrm{P}} = h(\dot{E}^{\mathrm{P}})^2 \tag{8.5}$$

We conclude that $\Delta\dot{\Sigma}\Delta\dot{E}^{\mathrm{P}} > 0$.

(iii) $\dot{E}^{\mathrm{P}} = 0$, $\dot{E}^{\mathrm{P}^*} > 0$ (i.e. $\dot{\Lambda} = 0$, $\dot{\Lambda}^* > 0$). Then, $\dot{\Sigma}_Y = 0$ and $\dot{\Sigma} \leqslant 0$. Consequently

$$\Delta\dot{\Sigma}\Delta\dot{E}^{\mathrm{P}} \geqslant \dot{\Sigma}^*\dot{E}^{\mathrm{P}^*} = h(\dot{E}^{\mathrm{P}^*})^2 > 0 \tag{8.6}$$

(iv) $\dot{E}^{\mathrm{P}} = 0$, $\dot{E}^{\mathrm{P}^*} = 0$ (i.e. $\dot{\Lambda} = 0$, $\dot{\Lambda}^* = 0$). Then, $\Delta\dot{\Sigma}\Delta\dot{E}^{\mathrm{P}} = 0$.

We conclude that $\Delta\dot{r}_I\Delta\dot{e}_I = \Delta\dot{\Sigma}\Delta\dot{E}^{\mathrm{P}} \geqslant 0$. This argument can be repeated for the interface terms to obtain $\Delta\dot{s}_\alpha\Delta\dot{d}_\alpha = \Delta\dot{S}\Delta\dot{D} \geqslant 0$.

The work statement (8.1) implies that the elastic strain rate $\dot{\varepsilon}^{\mathrm{EL}}_{kl}$ and the plastic multipliers $(\dot{E}^{\mathrm{P}}, \dot{D})$ are unique. Consequently, the stress rate $\dot{\sigma}_{ij}$ as prescribed by Eq. (5.3) is unique, and $(\dot{\boldsymbol{e}}, \dot{\boldsymbol{d}})$ are unique via Eqs. (5.1) and (5.4).

9. Minimum principle II to obtain the plastic multipliers

Assume that the stress $\sigma_{ij}(\boldsymbol{x})$ is known everywhere, and $(\boldsymbol{r}, \boldsymbol{s})$ is known in the active plastic zone upon making use of minimum principle I. Write $(\dot{E}^{\mathrm{P}}, \dot{D})$ within each active plastic zone as $\dot{E}^{\mathrm{P}} \equiv \dot{\Lambda}\hat{E}$ and $\dot{D} \equiv \dot{\Lambda}\hat{D}$, upon introducing the normalized fields $(\hat{E}, \hat{D})$ of 'unit magnitude'. We emphasise that the plastic multiplier $\dot{\Lambda}$ is constant within each active plastic zone, but as yet undetermined. The plastic strain rate and its gradient are known from Eqs. (5.1) and (5.4). Now construct the incremental boundary value problem. Consider the functional

$$\begin{aligned} J(\dot{u}^*_i, \dot{\Lambda}^*) = {} & \frac{1}{2}\int_V \{L_{ijkl}(\dot{\varepsilon}^*_{ij} - \dot{\varepsilon}^{\mathrm{PL}^*}_{ij})(\dot{\varepsilon}^*_{kl} - \dot{\varepsilon}^{\mathrm{PL}^*}_{kl}) + h\dot{E}^{\mathrm{P}^{*2}}\}\mathrm{d}V \\ & + \frac{1}{2}\int_\Gamma \{g\dot{D}^{*2}\}\mathrm{d}\Gamma - \int_{S_T}\{\dot{T}^0_i\dot{u}^*_i + \dot{t}^0\dot{\varepsilon}^{\mathrm{P}^*}\}\mathrm{d}S \end{aligned} \tag{9.1}$$

We shall show that this can be minimized over all $(\dot{u}^*_i, \dot{\Lambda}^* \geqslant 0)$ to deliver the actual solution $(\dot{u}_i, \dot{\Lambda})$. $\dot{\sigma}_{ij}$ is the actual, equilibrium stress rate associated with $(\dot{u}_i, \dot{E}^{\mathrm{P}}, \dot{D})$.

Proof. The proof requires several convexity statements. As shorthand, write

$$V^{\mathrm{EL}}(\dot{\varepsilon}^{\mathrm{EL}}_{ij}) \equiv \frac{1}{2}L_{ijkl}\dot{\varepsilon}^{\mathrm{EL}}_{ij}\dot{\varepsilon}^{\mathrm{EL}}_{kl}, \quad V^{\mathrm{PL}}(\dot{E}^{\mathrm{P}}) \equiv \frac{1}{2}h\left(\dot{E}^{\mathrm{P}}\right)^2 \text{ and } W^{\mathrm{PL}}(\dot{D}) \equiv \frac{1}{2}g\dot{D}^2 \tag{9.2}$$

Then, Eq. (9.1) can be re-written as

$$J(\dot{u}_i, \dot{E}^{\mathrm{P}}, \dot{D}) = \int_V \{V^{\mathrm{EL}}(\dot{\varepsilon}^{\mathrm{EL}}_{ij}) + V^{\mathrm{PL}}(\dot{E}^{\mathrm{P}})\}\mathrm{d}V + \int_\Gamma \{W^{\mathrm{PL}}(\dot{D})\}\mathrm{d}\Gamma - \int_{S_T}\{\dot{T}^0_i\dot{u}_i + \dot{t}^0\dot{\varepsilon}^{\mathrm{P}}\}\mathrm{d}S \tag{9.3}$$

Now consider the trial solution $(\dot{u}^*_i, \dot{E}^{\mathrm{P}^*}, \dot{D}^*)$ in addition to the actual solution $(\dot{u}_i, \dot{E}^{\mathrm{P}}, \dot{D})$ and write $\Delta\dot{u}_i \equiv \dot{u}^*_i - \dot{u}_i$, $\Delta\dot{E}^{\mathrm{P}} \equiv \dot{E}^{\mathrm{P}^*} - \dot{E}^{\mathrm{P}}$ and $\Delta\dot{D} \equiv \dot{D}^* - \dot{D}$. Then, since $V^{\mathrm{EL}}(\dot{\varepsilon}^{\mathrm{EL}}_{ij})$, $V^{\mathrm{PL}}(\dot{E}^{\mathrm{P}})$ and $W^{\mathrm{PL}}(\dot{D})$ are strictly convex in their arguments, we have

$$\Delta V^{\mathrm{EL}} \geqslant \frac{\partial V^{\mathrm{EL}}}{\partial\dot{\varepsilon}^{\mathrm{EL}}_{ij}}\Delta\dot{\varepsilon}^{\mathrm{EL}}_{ij} = L_{ijkl}\dot{\varepsilon}^{\mathrm{EL}}_{kl}\Delta\dot{\varepsilon}^{\mathrm{EL}}_{ij} \tag{9.4}$$

$$\Delta V^{\mathrm{PL}}(\dot{E}^{\mathrm{P}}) \geqslant \frac{\partial V^{\mathrm{PL}}}{\partial\dot{E}^{\mathrm{P}}}\Delta\dot{E}^{\mathrm{P}} = h\dot{E}^{\mathrm{P}}\Delta\dot{E}^{\mathrm{P}} = hA_{IJ}\dot{e}_I\Delta\dot{e}_J \tag{9.5}$$

and

$$\Delta W^{\mathrm{PL}}(\dot{D}) \geqslant \frac{\partial W^{\mathrm{PL}}}{\partial \dot{D}} \Delta \dot{D} = g\dot{D}\Delta\dot{D} = g a_{\alpha\beta}\dot{d}_\alpha \Delta \dot{d}_\beta \tag{9.6}$$

It is understood that the inequalities become equalities when the two solutions coincide.

Also note that

$$\int_{S_T} \dot{T}_i \Delta \dot{u}_i \,\mathrm{d}S = \int_S \dot{T}_i \Delta \dot{u}_i \,\mathrm{d}S = \int_V \dot{\sigma}_{ij} \Delta \dot{\varepsilon}_{ij} \,\mathrm{d}V \tag{9.7}$$

and

$$\begin{aligned} \int_{S_T} \{\dot{t}\Delta\dot{\varepsilon}^{\mathrm{P}}\}\mathrm{d}S &= \int_S \{\dot{t}\Delta\dot{\varepsilon}^{\mathrm{P}}\}\mathrm{d}S = \int_S \{\dot{\tau}_i n_i \Delta\dot{\varepsilon}^{\mathrm{P}}\}\mathrm{d}S \\ &= \int_V \{(\dot{Q} - \dot{\sigma}_e)\Delta\dot{\varepsilon}^{\mathrm{P}} + \dot{\tau}_i \Delta\dot{\varepsilon}^{\mathrm{P}}_{,i}\}\mathrm{d}V + \int_\Gamma \{n_i[\dot{\tau}_i \Delta\dot{\varepsilon}^{\mathrm{P}}]\}\mathrm{d}\Gamma \end{aligned} \tag{9.8}$$

Now proceed to evaluate

$$\Delta J \equiv J(\dot{u}_i^*, \dot{\Lambda}^*) - J(\dot{u}_i, \dot{\Lambda}) \tag{9.9}$$

Making use of Eqs. (9.4)–(9.8) and the rate form of Eqs. (2.8)–(2.10) allows Eq. (9.9) to be reduced to

$$\Delta J \geqslant \int_V \{(hA_{IJ}\dot{e}_J - \dot{r}_I)\Delta\dot{e}_I\}\mathrm{d}V + \int_\Gamma \{(g a_{\alpha\beta}\dot{d}_\beta - \dot{s}_\alpha)\Delta\dot{d}_\alpha\}\mathrm{d}\Gamma \tag{9.10}$$

We proceed to demonstrate that the r.h.s. of Eq. (9.10) is non-negative by proving that $\dot{r}_I\Delta\dot{e}_I \leqslant hA_{IJ}\dot{e}_J\Delta\dot{e}_I$ and $\dot{s}_\alpha\Delta\dot{d}_\alpha \leqslant g a_{\alpha\beta}\dot{d}_\beta\Delta\dot{d}_\alpha$ at each point. First, we show that $\dot{r}_I\Delta\dot{e}_I \leqslant hA_{IJ}\dot{e}_J\Delta\dot{e}_I$ by considering the four distinct cases:

(i) $\dot{E}^{\mathrm{P}} > 0$, $\dot{E}^{\mathrm{P}^*} > 0$ (i.e. $\dot{\Lambda} > 0$, $\dot{\Lambda}^* > 0$). Then, $\dot{r}_I\Delta\dot{e}_I = hA_{IJ}\dot{e}_J\Delta\dot{e}_I$.

(ii) $\dot{E}^{\mathrm{P}} > 0$, $\dot{E}^{\mathrm{P}^*} = 0$ (i.e. $\dot{\Lambda} > 0$, $\dot{\Lambda}^* = 0$).
Then, $\dot{r}_I\Delta\dot{e}_I = -\dot{r}_I\dot{e}_I = -hA_{IJ}\dot{e}_J\dot{e}_I = hA_{IJ}\dot{e}_J\Delta\dot{e}_I$.

(iii) $\dot{E}^{\mathrm{P}} = 0$, $\dot{E}^{\mathrm{P}^*} > 0$ (i.e. $\dot{\Lambda} = 0$, $\dot{\Lambda}^* > 0$). Then, $\dot{e}_I = 0$, $r_I\dot{r}_I < 0$ and $\Delta\dot{e}_I = \dot{e}_I^* = \dot{E}^{\mathrm{P}^*} r_I/\Sigma$.
Consequently, $\dot{r}_I\Delta\dot{e}_I = r_I\dot{r}_I(\dot{E}^{\mathrm{P}^*}/\Sigma) < 0$. Also, $hA_{IJ}\dot{e}_J\Delta\dot{e}_I = 0$ since $\dot{e}_I = 0$. We conclude that $\dot{r}_I\Delta\dot{e}_I < hA_{IJ}\dot{e}_J\Delta\dot{e}_I$ in this case.

(iv) $\dot{E}^{\mathrm{P}} = 0$, $\dot{E}^{\mathrm{P}^*} = 0$. (i.e. $\dot{\Lambda} = 0$, $\dot{\Lambda}^* = 0$). Then, $\Delta\dot{e}_I = 0$ and so $\dot{r}_I\Delta\dot{e}_I = hA_{IJ}\dot{e}_J\Delta\dot{e}_I = 0$.

A similar argument can be used to prove that $\dot{s}_\alpha\Delta\dot{d}_\alpha \leqslant g a_{\alpha\beta}\dot{d}_\beta\Delta\dot{d}_\alpha$. Therefore, the r.h.s. of Eq. (9.10) is non-negative and the minimum principle is established.

Further manipulation of Eq. (9.1) using Eqs. (5.2) and (5.5) allows ΔJ in Eq. (9.10) to be re-expressed as

$$\Delta J \geqslant \int_V \{(h\dot{E}^{\mathrm{P}} - \dot{\Sigma})\Delta\dot{E}^{\mathrm{P}}\}\mathrm{d}V + \int_\Gamma \{(g\dot{D} - \dot{S})\Delta\dot{D}\}\mathrm{d}\Gamma \tag{9.11}$$

Straightforward manipulations can be used to show that at the solution

$$2J_{\min} = \int_{S_U} \{\dot{T}_i\dot{u}_i^0 + \dot{t}\dot{\varepsilon}^{\mathrm{P0}}\}\mathrm{d}S - \int_{S_T} \{\dot{T}_i^0\dot{u}_i + \dot{t}^0\dot{\varepsilon}^{\mathrm{P}}\}\mathrm{d}S \tag{9.12}$$

A finite element implementation of minimum principle II is included in Appendix A.

9.1. Dual formulation

A dual formulation exists to (9.1). Write $M_{ijkl} \equiv (L^{-1})_{ijkl}$ as the elastic compliance, and introduce the functional

$$F(\dot{\sigma}_{ij}^*, \dot{r}_I^*, \dot{s}_\alpha^*) \equiv \frac{1}{2}\int_V \left\{ M_{ijkl}\dot{\sigma}_{ij}^*\dot{\sigma}_{kl}^* + \frac{1}{h}\left(\frac{D_{IJ}r_J\dot{r}_I^*}{\Sigma}\right)^2 \right\}\mathrm{d}V + \int_\Gamma \left\{ \frac{1}{2g}\left(\frac{b_{\alpha\beta}s_\beta\dot{s}_\alpha^*}{S}\right)^2 \right\}\mathrm{d}\Gamma - \int_{S_U} \{\dot{T}_i^*\dot{u}_i^0 + \dot{t}^*\dot{\varepsilon}^{\mathrm{P0}}\}\mathrm{d}S \tag{9.13}$$

Then, $F(\dot{\sigma}_{ij}^*, \dot{r}_I^*, \dot{s}_\alpha^*)$ is minimized over all equilibrium fields in order to deliver the actual solution. The proof begins with the principle of virtual work (2.4) written in the form

$$\int_S \{\Delta\dot{T}_i\dot{u}_i + \Delta\dot{t}\dot{\varepsilon}^{\mathrm{P}}\}\mathrm{d}S = \int_V \{\Delta\dot{\sigma}_{ij}\dot{\varepsilon}_{ij}^{\mathrm{EL}} + \Delta\dot{r}_I\dot{e}_I\}\mathrm{d}V + \int_\Gamma \{\Delta\dot{s}_\alpha\dot{d}_\alpha\}\mathrm{d}\Gamma \tag{9.14}$$

where the symbol Δ again denotes the difference between the trial equilibrium field solution (asterisked) and the actual solution (no asterisk). Now note that

$$\Delta\dot{\sigma}_{ij}\dot{\varepsilon}_{ij}^{\mathrm{EL}} \leqslant \frac{1}{2}M_{ijkl}\dot{\sigma}_{ij}^*\dot{\sigma}_{kl}^* - \frac{1}{2}M_{ijkl}\dot{\sigma}_{ij}\dot{\sigma}_{kl} \tag{9.15}$$

Also, for $\dot{E}^{\mathrm{P}} > 0$ we have

$$\Delta \dot{r}_I \dot{e}_I \leqslant \frac{1}{2h}\left(\frac{D_{IJ} r_J \dot{r}_I^*}{\Sigma}\right)^2 - \frac{1}{2h}\left(\frac{D_{IJ} r_J \dot{r}_I}{\Sigma}\right)^2 \tag{9.16}$$

and for $\dot{D} > 0$

$$\Delta \dot{s}_\alpha \dot{d}_\alpha \leqslant \frac{1}{2g}\left(\frac{b_{\alpha\beta} s_\beta \dot{s}_\alpha^*}{S}\right)^2 - \frac{1}{2g}\left(\frac{b_{\alpha\beta} s_\beta \dot{s}_\alpha}{S}\right)^2 \tag{9.17}$$

Substitution of Eqs. (9.15)–(9.17) into Eq. (9.14), and suitable re-arrangement of terms leads to the result

$$F(\dot{\sigma}_{ij}, \dot{r}_I, \dot{s}_\alpha) \leqslant F(\dot{\sigma}_{ij}^*, \dot{r}_I^*, \dot{s}_\alpha^*) \tag{9.18}$$

This completes the proof. Strictly, in Eq. (9.13), the volume integral for the term $(D_{IJ} r_J \dot{r}_I^*/\Sigma)^2/h$ and the surface integral for the term $(b_{\alpha\beta} s_\beta \dot{s}_\alpha^*/S)^2/g$ should be performed only over the active plastic zone, which in general is not known in advance. Upon performing the integrals everywhere, the bound (9.13) remains true but is weakened. Finally, we note that $F(\dot{\sigma}_{ij}, \dot{r}_I, \dot{s}_\alpha)$ can be written in the form:

$$-2F(\dot{\sigma}_{ij}, \dot{r}_I, \dot{s}_\alpha) = \int_{S_U} \{\dot{T}_i \dot{u}_i^0 + \dot{t}\dot{\varepsilon}^{\mathrm{P}0}\}\mathrm{d}S - \int_{S_T} \{\dot{T}_i^0 \dot{u}_i + \dot{t}^0 \dot{\varepsilon}^{\mathrm{P}}\}\mathrm{d}S \tag{9.19}$$

and so we conclude that the values of $2J$ in the extremum of Eq. (9.1) and of $-2F$ in the extremum of Eq. (9.13) approximate the quantity $\int_{S_U}\{\dot{T}_i \dot{u}_i^0 + \dot{t}\dot{\varepsilon}^{\mathrm{P}0}\}\mathrm{d}S - \int_{S_T}\{\dot{T}_i^0 \dot{u}_i + \dot{t}^0 \dot{\varepsilon}^{\mathrm{P}}\}\mathrm{d}S$ from above and below.

10. Case study: shearing of a layer sandwiched between two substrates

It is instructive to use the above flow theory formulation to examine the size effect anticipated for an infinite elastic–plastic layer, $-\infty < x_1 < \infty$, of height $2L$, with $-L < x_2 < L$. Each face is bonded to a rigid substrate, and a shear traction σ_{12}^∞ is applied to the layer via the rigid substrates; consequently, the top and bottom of the layer respond with displacements $u_1 = U$ and $u_1 = -U$, respectively. A Ramberg–Osgood curve is taken to characterize the uniaxial tensile stress-strain curve of the solid. Specifically

$$\varepsilon/\varepsilon_0 = \sigma/\sigma_0 + (\sigma/\sigma_0)^n \tag{10.1}$$

where $\varepsilon_0 = \sigma_0/E$ and E is Young's modulus. For this choice, the hardening modulus h reads $h(E^{\mathrm{P}}) = NE(E^{\mathrm{P}}/\varepsilon_0)^{N-1}$ with $N = 1/n$.

The shear-layer problem has been addressed previously by Fleck and Hutchinson (2001), and we shall show that the revised theory delivers identical predictions. This arises because the boundary value problem considered gives rise to proportional plastic straining at each material point: consequently, the two theories coincide, as proved formally below in the concluding remarks of this paper. It remains instructive to address the sandwich layer problem using the current formulation in order to demonstrate how the theory can be used in practice.

The only non-zero displacement $u_1(x_2)$ is assumed to be independent of x_1. Consequently, the only non-zero strain quantity is the shear strain $\gamma(x_2) = \mathrm{d}u_1/\mathrm{d}x_2$. With $\gamma^{\mathrm{P}}(x_2)$ denoting the plastic shear strain, the von Mises plastic strain rate $\dot{\varepsilon}^{\mathrm{P}}$ reads $\dot{\varepsilon}^{\mathrm{P}} = |\dot{\gamma}^{\mathrm{P}}|/\sqrt{3}$ and the overall effective plastic strain rate $\dot{E}^{\mathrm{P}}$, as defined in Eq. (3.4), reduces to

$$\left(\dot{E}^{\mathrm{P}}\right)^2 = \frac{1}{3}(\dot{\gamma}^{\mathrm{P}})^2 + \frac{1}{3}\ell^2\left(\frac{\mathrm{d}\dot{\gamma}^{\mathrm{P}}}{\mathrm{d}x_2}\right)^2 \tag{10.2}$$

where

$$\ell^2 = \frac{4}{15}\ell_1^2 + \frac{1}{3}\ell_2^2 + \frac{2}{5}\ell_3^2 \tag{10.3}$$

in the three length-scale version and $\ell = \ell^*$ in the single length-scale version of the theory.

These connections have previously been stated as (33) and (34) of Fleck and Hutchinson (2001).

Assume that the plastic strain rate vanishes at the top and bottom of the layer. The minimum principle I (7.1) then specializes to

$$H = \inf_{(\dot{\varepsilon}^{\mathrm{P}*} \geqslant 0)} 2\int_0^L \{\Sigma_Y \dot{E}^{\mathrm{P}^*} - \sigma_e \dot{\varepsilon}^{\mathrm{P}^*}\}\mathrm{d}x_2 \tag{10.4}$$

Upon taking the infimum (10.4) delivers

$$\int_0^L \left\{\Sigma_Y \frac{\dot{\varepsilon}^{\mathrm{P}}}{\dot{E}^{\mathrm{P}}}\delta\dot{\varepsilon}^{\mathrm{P}} + \Sigma_Y \ell^2 \frac{\dot{\varepsilon}_{,2}^{\mathrm{P}}}{\dot{E}^{\mathrm{P}}}\delta\dot{\varepsilon}_{,2}^{\mathrm{P}} - \sigma_e \delta\dot{\varepsilon}^{\mathrm{P}}\right\}\mathrm{d}x_2 = 0 \tag{10.5}$$

Now introduce

$$Q = \Sigma_Y \frac{\dot{\varepsilon}^{\mathrm{P}}}{\dot{E}^{\mathrm{P}}} \text{ and } \tau_2 = \Sigma_Y \ell^2 \frac{\dot{\varepsilon}^{\mathrm{P}}_{,2}}{\dot{E}^{\mathrm{P}}} \tag{10.6}$$

consistent with (4.3), and integrate (10.5) by parts to obtain the equilibrium relation

$$Q - \tau_{2,2} - \sigma_e = 0 \text{ within } V \tag{10.7}$$

along with the boundary condition that $\delta\dot{\varepsilon}^{\mathrm{P}} \equiv 0$ on the top and bottom of the layer.

Now proceed to consider minimum principle II. The functional (9.1) simplifies to

$$J(\dot{u}_i^*, \dot{E}^{\mathrm{P}^*}) = \int_0^L \{G(\dot{\gamma}^* - \dot{\gamma}^{\mathrm{P}^*})^2 + h(\dot{E}^{\mathrm{P}^*})^2\} \mathrm{d}x_2 \tag{10.8}$$

where $G = \sigma_0/(3\varepsilon_0)$ is the usual elastic shear modulus. Define the shear stress σ_{12} by $\sigma_{12} = G(\gamma^* - \gamma^{\mathrm{P}^*})$ and the effective stress Σ via

$$\Sigma^2 = Q^2 + \ell^{-2}(\tau_2)^2 \tag{10.9}$$

Then minimize the functional J with respect to the trial fields $(\dot{u}_i^*, \dot{E}^{\mathrm{P}^*})$ to obtain the Euler relations within V:

$$\dot{\sigma}_{12,2} = 0 \quad \text{and} \quad \dot{\Sigma} = h\dot{E}^{\mathrm{P}} \tag{10.10}$$

(The second equation of (10.10) assumes that active yield is taking place.) We conclude that the shear stress is spatially uniform, and is related to the von Mises effective stress via $\sigma_e = |\sigma_{12}|\sqrt{3}$.

The above system of equations can be reduced to a single non-linear ordinary differential equation (o.d.e.) as follows. Substitute (10.6) into (10.7) to obtain

$$-\left(\Sigma_Y \ell^2 \frac{\dot{\varepsilon}^{\mathrm{P}}_{,2}}{\dot{E}^{\mathrm{P}}}\right)_{,2} + \Sigma_Y \frac{\dot{\varepsilon}^{\mathrm{P}}}{\dot{E}^{\mathrm{P}}} = |\sigma_{12}|\sqrt{3} \tag{10.11}$$

Alternatively, the solution for an incremental step of shear traction can be obtained by re-writing the above equations as a system of three o.d.e's in the incremental quantities $(\dot{\varepsilon}^{\mathrm{P}}, \dot{\tau}_2, \dot{u}_1)$, giving

$$\frac{\mathrm{d}\dot{\varepsilon}^{\mathrm{P}}}{\mathrm{d}x_2} = \frac{\tau_2}{\ell^2 Q}\dot{\varepsilon}^{\mathrm{P}} \tag{10.12a}$$

$$\frac{\mathrm{d}\dot{\tau}_2}{\mathrm{d}x_2} = \left(\frac{\Sigma}{Q}\right)^2 h\dot{\varepsilon}^{\mathrm{P}} - \frac{\tau_2}{\ell^2 Q}\dot{\tau}_2 - |\dot{\sigma}_{12}|\sqrt{3} \tag{10.12b}$$

and

$$\frac{\mathrm{d}|\dot{u}_1|}{\mathrm{d}x_2} = \dot{\varepsilon}^{\mathrm{P}}\sqrt{3} + \frac{|\dot{\sigma}_{12}^{\infty}|}{G} \tag{10.12c}$$

This two-point boundary value problem has been solved numerically using the NAG Fortran module D02GAF[1]: it entails a finite difference technique with deferred correction and Newton iteration. An alternative approach would be to perform a finite element discretisation of the functional in Eq. (10.4) and subsequent minimization with respect to the nodal amplitudes. This approach was used by Fleck and Hutchinson (2001) but is not pursued further here.

10.1. Results

The shear traction versus displacement jump across the sandwich layer is presented in Fig. 2 for the choice $\ell/L = 0$, 0.5 and strain hardening exponent $n = 5$. In each case, the layer was loaded until the displacement attained the value $(U/\varepsilon_0 L) = 9.6$. The traction rate was then reversed in sign and the layer was reverse loaded until $(U/\varepsilon_0 L)$ attained the value of zero. In the case of $\ell/L = 0$, conventional isotropic J_2 flow theory is recovered and the plastic strain is uniform across the layer. In contrast, a boundary layer in γ^{P} exists for finite ℓ/L, and this is shown explicitly in Fig. 3 at peak load, labelled state A in Fig. 2, and upon full reversal of displacement, labelled state B. The strain-gradient theory is able to capture reversed plastic loading, and an isotropic hardening response is observed.

10.2. Comparison with the Fleck and Hutchinson (2001) theory

It is instructive to compare the governing o.d.e. (10.11) with that obtained by Fleck and Hutchinson (2001). They obtained

$$-\ell^2(h(E_{\mathrm{P}})\dot{\varepsilon}^{\mathrm{P}}_{,2})_{,2} + h(E_{\mathrm{P}})\dot{\varepsilon}^{\mathrm{P}} = |\dot{\sigma}_{12}|\sqrt{3} \tag{10.13}$$

[1] Release 19 of the Numerical Algorithms Group Ltd., Wilkinson House, Jordan Hill Rd., Oxford OX2 8DR, UK.

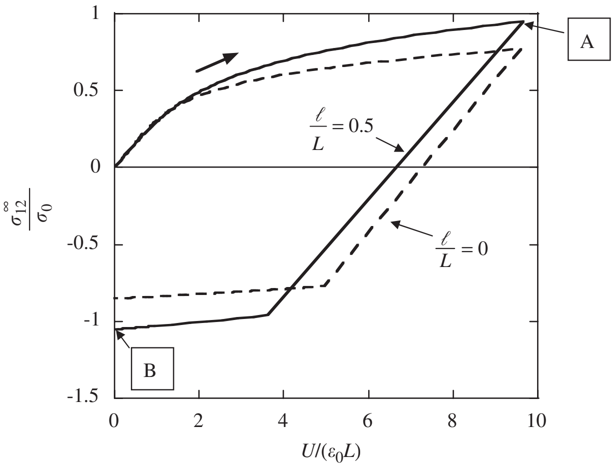

Fig. 2. Shear traction versus displacement jump across a sandwiched plastic layer, for the strain-gradient solid with $\ell/L = 0.5$ and for the conventional J_2 flow theory solid, $\ell/L = 0$. In each case the direction of loading is reversed when the displacement jump across the layer has attained the value $(U/\varepsilon_0 L) = 9.6$. For the choice $\ell/L = 0.5$, the state at peak load is labelled A, while the state $(U/\varepsilon_0 L) = 0$ after load reversal is labelled B.

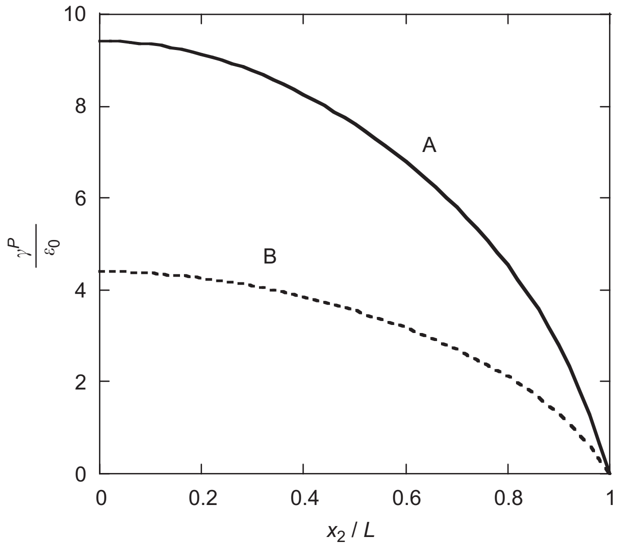

Fig. 3. The plastic shear strain distribution within the strain-gradient solid, $\ell/L = 0.5$, for states A and B. These states have already been defined in Fig. 2.

as stated in (37) of their paper. In general, the two governing equations are different. However, for the case of proportional plastic straining (10.11) reduces to (10.13), as seen by time differentiation of both sides of (10.11). The solution to the shear-layer problem (10.11) satisfies proportional plastic straining and so the two theories coincide for this problem, including the case of reversed plastic loading.

The Fleck and Hutchinson (2001) formulation is best contrasted with that of the current theory by comparing their respective flow laws. Fleck and Hutchinson (2001) give

$$\frac{\dot{\varepsilon}^{\mathrm{P}}_{,2}}{\dot{\varepsilon}^{\mathrm{P}}} = \frac{\dot{\tau}_2}{\ell^2 \dot{Q}} \tag{10.14}$$

while the current formulation gives via (10.6)

$$\frac{\dot{\varepsilon}^{\mathrm{P}}_{,2}}{\dot{\varepsilon}^{\mathrm{P}}} = \frac{\tau_2}{\ell^2 Q} \tag{10.15}$$

It is clear from these two equations that the respective theories give identical predictions in the case of proportional plastic straining.

11. Concluding remarks

The above rate-independent flow theory of gradient plasticity has close connections to that given previously by Fleck and Hutchinson (2001): in the absence of interfaces, the field equations are the same, whereas the yield condition and flow law are different. In fact, in any case of proportional plastic straining (including reversed loading) the present theory coincides with that of Fleck and Hutchinson (2001). The proof of this is straightforward and is outlined as follows.

Fleck and Hutchinson (2001) did not develop a constitutive law for yielding at an interface, and so we restrict attention here to the case where interfaces are absent from the body. Recall that the current theory relates the generalized stresses r_I to the generalized strain rate $\dot{e}_I$ according to Eq. (4.3), with $\dot{E}^{\mathrm{P}}$ given by Eq. (3.1). Now assume that proportional plastic straining occurs, such that

$$\dot{\varepsilon}^{\mathrm{P}}(\boldsymbol{x},t) = f(\boldsymbol{x})g(t) \tag{11.1}$$

in terms of arbitrary functions $f(\boldsymbol{x})$ and $g(t)$. Substitute Eq. (11.1) into Eqs. (4.3) and (3.1), and differentiate the relation obtained with respect to time. Then, use of $\dot{\Sigma} = h\dot{E}^{\mathrm{P}}$ leads to

$$\dot{r}_I = hA_{IJ}\dot{e}_J \tag{11.2}$$

This relation is identical to the assumed flow law of Fleck and Hutchinson (2001), Eqs. (22) and (23).

Proportional plastic straining in the current flow theory leads directly to proportional stressing and to proportional elastic straining. In the case of continually increasing plastic strain $g(t)>0$ for all t, the current flow theory also coincides with the deformation theory version of gradient plasticity as laid down by Fleck and Hutchinson (2001).

The above development assumes a particular form for the dissipation potentials and flow rule for practical implementation without undue complexity. A more complete development, including an allowance for energetic stresses and dissipation at interfaces, will be presented in Part II. In this subsequent development, the plastic strain rate $\dot{\varepsilon}^{\mathrm{PL}}_{ij}$ is treated as a free kinematic variable, as was done by Fleck and Willis (2004) in the context of deformation theory.

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Appendix A. Finite element implementation of the minimum principles

A brief outline is now given of how the two minimum principles might be implemented in the finite element context.

A.1. Minimum principle I

There are two distinct cases to consider:

(i) If we know $\dot{\varepsilon}^{\mathrm{P}}$ on a portion of S_u at the boundary of an active plastic zone, then we can use minimum principle I to obtain $\dot{\Lambda}_{\mathrm{a}}$ and $\hat{\varepsilon}(\boldsymbol{x})$ for this plastic zone.
(ii) If $\dot{\varepsilon}^{\mathrm{P}}$ is not prescribed at the boundary of an active plastic zone, then minimum principle I delivers $\hat{\varepsilon}(\boldsymbol{x})$ but not $\dot{\Lambda}_{\mathrm{a}}$ for this plastic zone. We consider this case in more detail as follows.

Write the normalized plastic strain rate $\hat{\varepsilon}(\boldsymbol{x})$ at node J of a finite element mesh by $a^{(J)}\geqslant 0$ and introduce the scalar shape functions $M^{(J)}(\boldsymbol{x})\geqslant 0$ such that the trial plastic strain rate $\hat{\varepsilon}(\boldsymbol{x})$ over the solid is

$$\hat{\varepsilon}(\boldsymbol{x}) = a^{(J)}M^{(J)}(\boldsymbol{x}), \quad \text{summed over nodes } J \tag{A.1}$$

Then, the functional to be minimized follows from Eq. (7.1) as

$$\hat{H}(\hat{\varepsilon}(\boldsymbol{x})) = \int_V \{\Sigma_Y\hat{E} - \sigma_e\hat{\varepsilon}\}\mathrm{d}V + \int_\Gamma \{S_Y\hat{D}\}\mathrm{d}\Gamma - \int_{S_T}\{t^0\hat{\varepsilon}\}\mathrm{d}S \tag{A.2}$$

where $(\hat{\varepsilon},\hat{E},\hat{D})$ are expressed in terms of the trial field $a^{(J)}$ according to (A.1). Additionally, impose the constraint (7.3) by introduction of a Lagrange multiplier λ_{a} for each active plastic zone, giving

$$\hat{H}(a^{(J)},\lambda_{\mathrm{a}}) = \int_V \{\Sigma_Y\hat{E} - \sigma_e\hat{\varepsilon}\}\mathrm{d}V + \int_\Gamma \{S_Y\hat{D}\}\mathrm{d}\Gamma - \int_{S_T}\{t^0\hat{\varepsilon}\}\mathrm{d}S - \frac{\lambda_{\mathrm{a}}}{V_{\mathrm{a}}}\int_{V_{\mathrm{a}}}[\hat{\varepsilon}-1]\mathrm{d}V_{\mathrm{a}} \tag{A.3}$$

The solution to (A.3) provides $\hat{\varepsilon}(\boldsymbol{x})$ uniquely.

A.2. Minimum principle II

Assume that minimum principle I has been invoked in order to solve for $\hat{\varepsilon}(\boldsymbol{x}) = a^{(J)}M^{(J)}(\boldsymbol{x})$. It remains to make use of minimum principle II in order to obtain $(\dot{u}_i, \dot{\Lambda}_{\rm a})$. To proceed, we introduce nodal values $b^{(J)}$ at each node J, and shape functions $N_i^{(J)}(\boldsymbol{x})$, for the velocity field $\dot{u}_i$ such that

$$\dot{u}_i = b^{(J)}N_i^{(J)}(\boldsymbol{x}), \quad \text{summed over nodes } J \tag{A.4}$$

Also, the plastic strain rate within each active plastic zone is written in terms of the known values $a^{(J)}$ and the scalar unknowns $\dot{\Lambda}_{\rm a}$ as

$$\dot{\varepsilon}^{\rm P}(\boldsymbol{x}) = \dot{\Lambda}_{\rm a}a^{(J)}M^{(J)}(\boldsymbol{x}) \tag{A.5}$$

Now substitute (A.4) and (A.5) into the functional (9.1) to obtain

$$J(b^{(J)}, \dot{\Lambda}_{\rm a}) = \frac{1}{2}\int_V \{L_{ijkl}(\dot{\varepsilon}^*_{ij} - \dot{\Lambda}_{\rm a}\hat{\varepsilon}_{ij})(\dot{\varepsilon}^*_{kl} - \dot{\Lambda}_{\rm a}\hat{\varepsilon}_{kl}) + h\dot{\Lambda}_{\rm a}^2\hat{E}^2\}{\rm d}V + \frac{1}{2}\int_\Gamma \{g\dot{\Lambda}_{\rm a}^2\hat{D}^2\}{\rm d}\Gamma - \int_{S_T}\{\dot{T}_i^0\dot{u}_i^* + \dot{t}^0\dot{\Lambda}_{\rm a}\hat{\varepsilon}\}{\rm d}S \tag{A.6}$$

The minimum value of $J(b^{(J)}, \dot{\Lambda}_{\rm a})$ is achieved by taking the first variation with respect to $(b^{(J)}, \dot{\Lambda}_{\rm a})$ such that the variational fields are

$$\delta\dot{u}_i = \delta b^{(J)}N_i^{(J)}(\boldsymbol{x}) \tag{A.7}$$

and

$$\delta\dot{\varepsilon}^{\rm P} = \delta\dot{\Lambda}_{\rm a}a^{(J)}M^{(J)}(\boldsymbol{x}) \tag{A.8}$$

A linear matrix equation follows directly of standard form

$$\begin{pmatrix} A^{(IJ)} & B_{\rm a}^{(J)} \\ B_{\rm a}^{(J)} & C_{\rm aa} \end{pmatrix}\begin{pmatrix} b^{(J)} \\ \dot{\Lambda}_{\rm a} \end{pmatrix} = \begin{pmatrix} F^{(I)} \\ G_{\rm a} \end{pmatrix} \tag{A.9}$$

with elements ($A^{(IJ)}$, $B_{\rm a}^{(J)}$, $C_{\rm aa}$, $F^{(I)}$, $G_{\rm a}$) all known from the finite element representation of (A.6). The algebraic system of Eq. (A.9) is solved in order to obtain $(b^{(J)}, \dot{\Lambda}_{\rm a})$ and thereby $(\dot{u}_i, \dot{\Lambda}_{\rm a})$.

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