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A mathematical basis for strain-gradient plasticity theory. Part II: Tensorial plastic multiplier

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ABSTRACT

A phenomenological, flow theory version of gradient plasticity for isotropic and anisotropic solids is constructed along the lines of Gudmundson [Gudmundson, P., 2004. A unified treatment of strain-gradient plasticity. J. Mech. Phys. Solids 52, 1379–1406]. Both energetic and dissipative stresses are considered in order to develop a kinematic hardening theory, which in the absence of gradient terms reduces to conventional J_2 flow theory with kinematic hardening. The dissipative stress measures, work-conjugate to plastic strain and its gradient, satisfy a yield condition with associated plastic flow. The theory includes interfacial terms: elastic energy is stored and plastic work is dissipated at internal interfaces, and a yield surface is postulated for the work-conjugate stress quantities at the interface. Uniqueness and extremum principles are constructed for the solution of boundary value problems, for both the rate-dependent and the rate-independent cases. In the absence of strain gradient and interface effects, the minimum principles reduce to the classical extremum principles for a kinematically hardening elasto-plastic solid. A rigid-hardening version of the theory is also stated and the resulting theory gives rise to an extension to the classical limit load theorems. This has particular appeal as previous trial fields for limit load analysis can be used to generate immediately size-dependent bounds on limit loads.

1. Introduction

Strain-gradient theories of plasticity have emerged in recent years, based on the underlying notion that strain gradients enhance dislocation density and thereby enhance material strength (see for example, Fleck and Hutchinson, 1997, 2001; Gao et al., 1999; Gurtin, 2000; Gudmundson, 2004). Both phenomenological and crystal versions have been developed, and have been implemented within a finite element context in order to predict a number of size effects: the Hall–Petch size effect, the bending and torsional strengths of micron-scale specimens and the indentation size effect (see for example, Aifantis and Willis, 2005, 2006; Hwang et al., 2002; Wei and Hutchinson, 2003; Kuroda et al., 2007; Engelen et al., 2006; Evans and Hutchinson, 2009). These higher order theories require higher order boundary conditions in order to satisfy the governing higher order partial differential equations. In broad terms, the mechanics literature has focussed more on prediction than on experimental evidence, and aspects of the underlying physics remain unresolved. For example, the relationship between the material length scale(s) entering the theories and the salient features of the dislocated state requires further investigation.

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Recently, Fleck and Willis (2009) (hereafter referred to as Part I) have synthesised a phenomenological strain-gradient theory which involves the scalar plastic multiplier $\dot{\varepsilon}^P$ as the primary kinematic unknown in addition to the displacement rate $\dot{u}_i$. Plastic work is taken to be purely dissipative in nature, and stress measures Q and τ_i are work-conjugate to $\dot{\varepsilon}^P$ and its spatial gradient $\dot{\varepsilon}^P_{,i}$, respectively. They postulate a quadratic yield surface in (Q, τ_i) space, and assume that $(\dot{\varepsilon}^P, \dot{\varepsilon}^P_{,i})$ is normal to it. This incremental plasticity theory assumes associated plastic flow and possesses a convex yield surface; consequently, the theory ensures positive plastic work. Uniqueness and extremum principles are stated and thereby give the theoretical underpinning for robust numerical solutions and homogenisation methods.

We emphasise that in Part I, and in Fleck and Hutchinson (2001), the plastic strain increment $\delta\varepsilon^{PL}_{ij}$ is taken to be co-directional with the deviatoric Cauchy stress $\sigma'_{ij} \equiv \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk}$. Here, we follow the strategy of Gudmundson (2004) and treat $\delta\varepsilon^{PL}_{ij}$ as a kinematically free but incompressible quantity and derive it from the constitutive law as specified below. In this manner, the current study extends Part I by considering the more general case of a strain-gradient theory in terms of a tensorial plastic strain rate $\dot{\varepsilon}^P_{ij}$. This more general theory is built upon the formulation of Gudmundson (2004) and establishes an explicit kinematic hardening theory, involving both dissipative and energetic stresses within the solid and at internal interfaces. The inclusion of interface terms is similar in concept to that outlined by Gudmundson (2004) and Aifantis and Willis (2005, 2006), but differs in its implementation through concentrating on flow theory and on plastic dissipation at interfaces. (Recently, Fredriksson and Gudmundson (2007) have explored the size effect associated with gradient plasticity within a thin film resting upon an elastic, visco-plastic interface.)

The higher order theory presented by Gudmundson (2004) is of similar type to the single crystal theories for single crystal plasticity and visco-plasticity of Gurtin (2000, 2002). A major thrust of the current paper is to outline the variational structure that accompanies this type of formulation, particularly in the rate-independent limit.

The structure of this paper is as follows. First, the virtual work statement and accompanying field equations are stated for the strain-gradient solid. The rate-dependent version and accompanying minimum principles are then given. The central body of the paper deals with the rate-independent solid, and uniqueness and extremum principles are stated. The rate independent, rigid-hardening case is also considered: the velocity field is the primary kinematic variable, and uniqueness theorems and extremum principles are stated. Size-dependent limit loads follow immediately, and in the absence of gradient terms, the classical limit load theorems are recovered.

2. Précis of the Gudmundson (2004) gradient theory

The thermodynamic framework and constitutive law are essentially those given in the work of Gudmundson (2004), which in turn draws heavily upon Fleck and Hutchinson (1997). The present treatment provides mathematical structure that accompanies this constitutive law, including uniqueness and minimum principles for rate-dependent and rate-independent versions. Connections are also made with conventional J_2 kinematic hardening theory, and rigid-hardening theory. First, we briefly outline the framework and we then extend the ideas for both rate-dependent behaviour and rate-independent plasticity.

The concepts have already been elaborated in Part I (Fleck and Willis, 2009) in the context of a scalar plastic strain ε^P and its gradient. The present work requires a similar development but employs the tensorial plastic strain measure ε^{PL}_{ij} as a free kinematic quantity in addition to the displacement u_i. The internal work statement involves increments in elastic strain $\delta\varepsilon^{EL}_{ij}$, plastic strain $\delta\varepsilon^{PL}_{ij}$ and plastic strain gradient $\delta\varepsilon^{PL}_{ij,k}$. Internal work is also expended at internal interfaces Γ such as grain boundaries, such that the displacement rate $\dot{u}_i$ is continuous across these interfaces but the plastic strain rate $\dot{\varepsilon}^{PL}_{ij}$ can jump in value. We adopt the sign convention that the normal to the interface n_i points the direction from the −ve side to the +ve side. The physical arguments given in the introduction of Part I suggest that $\dot{\varepsilon}^{PL}_{ij}$ on each side of the interface is best written in terms of the jump in value $[\dot{\varepsilon}^{PL}_{ij}] \equiv \dot{\varepsilon}^{PL+}_{ij} - \dot{\varepsilon}^{PL-}_{ij}$ across a point of Γ, along with the mean value $\langle\dot{\varepsilon}^{PL}_{ij}\rangle \equiv (\dot{\varepsilon}^{PL+}_{ij} + \dot{\varepsilon}^{PL-}_{ij})/2$. (Gudmundson (2004) did not make use of such a decomposition.)

In order to write the principle of virtual work, we introduce the symmetric Cauchy stress σ_{ij} and the general stress measures Q_{ij} and τ_{ijk} as work-conjugates to the increments $\delta\varepsilon^{EL}_{ij}$, $\delta\varepsilon^{PL}_{ij}$ and $\delta\varepsilon^{PL}_{ij,k}$, respectively, within the bulk. Likewise, p_{ij} and q_{ij} are work-conjugate to $[\delta\varepsilon^{PL}_{ij}]$ and $\langle\delta\varepsilon^{PL}_{ij}\rangle$, respectively, on internal interfaces. Then, for a solid of volume V and external surface S, the principle of virtual work reads

$$\int_V \left\{\sigma_{ij}\delta\varepsilon^{EL}_{ij} + Q_{ij}\delta\varepsilon^{PL}_{ij} + \tau_{ijk}\delta\varepsilon^{PL}_{ij,k}\right\} dV + \int_\Gamma \{p_{ij}[\delta\varepsilon^{PL}_{ij}] + q_{ij}\langle\delta\varepsilon^{PL}_{ij}\rangle\}\, d\Gamma = \int_S \{T_i\delta u_i + t_{ij}\delta\varepsilon^{PL}_{ij}\}\, dS \tag{2.1}$$

from which follow the governing equilibrium relations

$$\text{in } V: \quad \sigma_{ij,j} = 0 \text{ and } Q_{ij} - \tau_{ijk,k} - \sigma'_{ij} = 0 \tag{2.2}$$

$$\text{on } \Gamma: \quad p_{ij} = n_k\langle\tau_{ijk}\rangle \text{ and } q_{ij} = n_k[\tau_{ijk}] \tag{2.3}$$

$$\text{and on } S: \quad T_i = \sigma_{ij}n_j \text{ and } t_{ij} = \tau_{ijk}n_k \tag{2.4}$$

Following Gudmundson (2004), take the internal energy to be of the form $U = U(\varepsilon_{ij}^{EL}, \varepsilon_{ij}^{PL}, \varepsilon_{ij,k}^{PL})$ and thereby define the *energetic* stresses:

$$\sigma_{ij}^{E} = \frac{\partial U}{\partial \varepsilon_{ij}^{EL}}, \quad Q_{ij}^{E} = \frac{\partial U}{\partial \varepsilon_{ij}^{PL}} \text{ and } \tau_{ijk}^{E} = \frac{\partial U}{\partial \varepsilon_{ij,k}^{PL}} \tag{2.5}$$

The natural decomposition for U is

$$U = \tfrac{1}{2} L_{ijkl}(\varepsilon_{ij} - \varepsilon_{ij}^{PL})(\varepsilon_{kl} - \varepsilon_{kl}^{PL}) + U^{PL}(\varepsilon_{ij}^{PL}, \varepsilon_{ij,k}^{PL}) \tag{2.6}$$

in terms of a positive definite elastic modulus L_{ijkl}, giving the usual elasticity relation

$$\sigma_{ij}^{E} = L_{ijkl}(\varepsilon_{kl} - \varepsilon_{kl}^{PL}) \tag{2.7}$$

This form will be adopted in the sequel. It should perhaps be remarked that U^{PL} could, more generally, be taken as depending on the history of plastic strain and its gradient. It would be necessary only that, during an increment, $\delta U^{PL} = Q_{ij}^{E}\delta\varepsilon_{ij}^{PL} + \tau_{ijk}^{E}\delta\varepsilon_{ij,k}^{PL}$. As in Part I, our main concern is to develop a theory that can be applied with relative ease.

Likewise, a surface energy $U^{S} = U^{S}([\varepsilon_{ij}^{PL}], \langle \varepsilon_{ij}^{PL} \rangle)$ is introduced at internal interfaces Γ, giving rise to the energetic stress measures

$$p_{ij}^{E} = \frac{\partial U^{S}}{\partial [\varepsilon_{ij}^{PL}]} \text{ and } q_{ij}^{E} = \frac{\partial U^{S}}{\partial \langle \varepsilon_{ij}^{PL} \rangle} \tag{2.8}$$

We will not elaborate particular choices for the functions $U^{PL}(\varepsilon_{ij}^{PL}, \ \varepsilon_{ij,k}^{PL})$ and $U^{S}([\varepsilon_{ij}^{PL}], \langle \varepsilon_{ij}^{PL} \rangle)$ in the present paper, and leave this to future work. One possibility is to write these contributions to the free energy in terms of scalar invariants of the arguments, as has been done for a deformation theory of plasticity by Fleck and Willis (2004), and Aifantis and Willis (2005, 2006).

Dissipative stresses follow immediately

$$\text{in } V: \quad \sigma_{ij}^{D} \equiv \sigma_{ij} - \sigma_{ij}^{E}, \quad Q_{ij}^{D} \equiv Q_{ij} - Q_{ij}^{E}, \quad \tau_{ijk}^{D} \equiv \tau_{ijk} - \tau_{ijk}^{E} \tag{2.9}$$

$$\text{on } \Gamma: \quad p_{ij}^{D} = p_{ij} - p_{ij}^{E}, \quad q_{ij}^{D} = q_{ij} - q_{ij}^{E} \tag{2.10}$$

By the second law, for any physically allowed increment, the change in internal energy is less than or equal to the internal work, and consequently

$$\sigma_{ij}^{D}\delta\varepsilon_{ij}^{EL} + Q_{ij}^{D}\delta\varepsilon_{ij}^{PL} + \tau_{ijk}^{D}\delta\varepsilon_{ij,k}^{PL} \geqslant 0 \quad \text{in } V \tag{2.11}$$

and

$$p_{ij}^{D}[\delta\varepsilon_{ij}^{PL}] + q_{ij}^{D}\langle \delta\varepsilon_{ij}^{PL} \rangle \geqslant 0 \text{ on } \Gamma \tag{2.12}$$

Since (2.11) holds for any $\delta\varepsilon_{ij}^{EL}$ it follows that $\sigma_{ij}^{D} = 0$, giving $\sigma_{ij} = \sigma_{ij}^{E}$. The constitutive laws stated below satisfy (2.11) and (2.12) if $(\sigma_{ij}^{D}, \ \tau_{ijk}^{D})$ are related to a convex dissipation potential in $(\dot{\varepsilon}_{ij}^{PL}, \dot{\varepsilon}_{ij,k}^{PL})$, and if $(p_{ij}^{D}, \ q_{ij}^{D})$ are related to a convex dissipation potential in $([\dot{\varepsilon}_{ij}^{PL}], \langle \dot{\varepsilon}_{ij}^{PL} \rangle)$. A wide choice of dissipation potentials is possible. Here, we follow Gudmundson (2004) and Fleck and Hutchinson (1997), and assume that the plastic dissipation depends upon effective strain rates derived from quadratic forms within V and on Γ. The next step is to stipulate these effective strain rates.

3. The effective strain rates

Introduce an overall effective strain rate measure $\dot{E}^{P}$ in terms of $(\dot{\varepsilon}_{ij}^{PL}, \dot{\varepsilon}_{ij,k}^{PL})$ within V. The uniqueness and extremum principles given in the following sections can be derived for the case where $\dot{E}^{P}$ is a convex function, homogeneous and of degree one, in the strain rate and strain-gradient rate. However, to make the formulation more explicit, and to provide a prototypical working theory, we limit attention to a general quadratic form for $(\dot{E}^{P})^2$ in terms of $(\dot{\varepsilon}_{ij}^{PL}, \dot{\varepsilon}_{ij,k}^{PL})$. First, recall that $\dot{\varepsilon}_{ij}^{PL}$ is incompressible and symmetric and so it has five independent components. Consequently, its gradient $\dot{\varepsilon}_{ij,k}^{PL}$ has 15 independent components. Together, they can be expressed as a 20-dimensional vector $\dot{\boldsymbol{e}} = (\dot{e}_I) = (\dot{\varepsilon}_{ij}^{PL}, \dot{\varepsilon}_{ij,k}^{PL})$. Then, $(\dot{E}^{P})^2$ can be written symbolically as

$$(\dot{E}^{P})^2 \equiv A_{IJ}\dot{e}_I\dot{e}_J \quad (I, J) \text{ sum over } 1, 2, \ldots, 20 \tag{3.1}$$

in terms of the 20×20 symmetric and positive definite matrix A_{IJ}. This definition is sufficiently general that it can describe an anisotropic solid in addition to the isotropic case. The formulation simplifies considerably for the isotropic solid, as outlined by Fleck and Hutchinson (2001) and Smyshlyaev and Fleck (1996): in this case, $\dot{E}^{P}$ can be written in terms of the usual von Mises scalar invariant $\dot{\varepsilon}^P$ of the plastic strain rate $\dot{\varepsilon}^{PL}_{ij}$ and of the three invariants of $\dot{\varepsilon}^{PL}_{ij,k}$. The interested reader should refer to Smyshlyaev and Fleck (1996), Fleck and Hutchinson (2001), Gudmundson (2004) or Part I for the details.

Likewise, write $\dot{\boldsymbol{d}} = (\dot{d}_\alpha) = ([\dot{\varepsilon}^{PL}_{ij}], \langle\dot{\varepsilon}^{PL}_{ij}\rangle)$ as a 10-dimensional vector, defined at internal interfaces, and construct

$$\dot{D}^2 = a_{\alpha\beta}\dot{d}_\alpha\dot{d}_\beta \quad (\alpha,\beta)\text{ sum over }1,2,\ldots,10 \tag{3.2}$$

in terms of the 10×10 symmetric and positive definite matrix $(a_{\alpha\beta})$. We assume that the plastic dissipation depends directly upon $\dot{E}^P$ within V and upon $\dot{D}$ on Γ. A visco-plastic formulation is now given, followed by the rate-independent case. The development closely parallels that of Part I and Gudmundson (2004), and so we minimize the discussion here.

4. Visco-plastic version

4.1. Creep potential within V

The 20-dimensional vector $\boldsymbol{r} = (r_I) = (Q_{ij},\ \tau_{ijk})$ is work-conjugate to $\dot{e}_I$ within V, and is the sum of an energetic part $\boldsymbol{r}^E = (r_I{}^E) = (Q^E_{ij},\ \tau^E_{ijk})$ and a dissipative part $\boldsymbol{r}^D = (r_I{}^D) = (Q^D_{ij},\ \tau^D_{ijk})$. (More precisely, $\{(Q_{ij},\tau_{ijk})\}$ has 24 dimensions but the incompressible constraint permits restriction to the 20-dimensional subspace $\{(Q_{ij},\tau_{ijk}){:}Q_{jj}=0,\tau_{ijj}=0\}$. This point will not be laboured in the sequel.) A power-law creep theory is constructed from a creep potential $\phi(\dot{E}^P)$ within V as

$$\phi = \int r_I^D d\dot{e}_I \equiv \frac{\sigma_0\dot{\varepsilon}_0}{N+1}\left(\frac{\dot{E}^P}{\dot{\varepsilon}_0}\right)^{N+1} \tag{4.1}$$

where $(\sigma_0, \dot{\varepsilon}_0, N)$ are material constants in the usual notation for power-law creep and $\dot{E}^P$ has already been specified by (3.1). Define the scalar Σ as the work-conjugate to $\dot{E}^P$, such that

$$\Sigma = \frac{\partial\phi}{\partial\dot{E}^P} = \sigma_0\left(\frac{\dot{E}^P}{\dot{\varepsilon}_0}\right)^N \tag{4.2}$$

Then,

$$r_I^D = \frac{\partial\phi}{\partial\dot{e}_I} = \frac{\Sigma}{\dot{E}^P}A_{IJ}\dot{e}_J \tag{4.3}$$

and substitution into (3.1) provides the connection

$$\Sigma^2 = D_{IJ}r_I^D r_J^D \tag{4.4}$$

where $D_{IJ}\equiv(A^{-1})_{IJ}$. The plastic work-rate reads

$$r_I^D\dot{e}_I = \Sigma\dot{E}^P = \sigma_0\dot{\varepsilon}_0\left(\frac{\dot{E}^P}{\dot{\varepsilon}_0}\right)^{N+1} \tag{4.5}$$

and is non-negative as demanded by (2.11). The constitutive dual $\phi^*(\Sigma)$ of $\phi(\dot{E}^P)$ is

$$\phi^*(\Sigma) = \sup_{\dot{E}^P}(\Sigma\dot{E}^P - \phi(\dot{E}^P)) = \frac{N}{N+1}\sigma_0\dot{\varepsilon}_0\left(\frac{\Sigma}{\sigma_0}\right)^{(N+1)/N} \tag{4.6}$$

and $\dot{e}_I$ is related to $r_I{}^D$ according to

$$\dot{e}_I = \frac{\partial\phi^*}{\partial r_I^D} = \frac{\dot{E}^P}{\Sigma}D_{IJ}r_J^D \tag{4.7}$$

4.2. Creep potential on Γ

Introduce the 10-dimensional vector $\boldsymbol{s} = (s_\alpha) = (p_{ij},q_{ij})$ as the work-conjugate to $\dot{\boldsymbol{d}}$. As for the bulk stresses, $\boldsymbol{s}$ is decomposed additively into an energetic part $\boldsymbol{s}^E = (s^E_\alpha) = \left(p^E_{ij},\ q^E_{ijk}\right)$ and a dissipative part $\boldsymbol{s}^D = (s^D_\alpha) = \left(p^D_{ij},\ q^D_{ij}\right)$. The creep potential on each internal interface is also taken to be power-law in form:

$$\omega = \int \boldsymbol{s}^D\, d\dot{\boldsymbol{d}} \equiv \frac{\sigma'_0\dot{\varepsilon}'_0}{M+1}\left(\frac{\dot{D}}{\dot{\varepsilon}'_0}\right)^{M+1} \tag{4.8}$$

where $(\sigma'_0, \dot{\varepsilon}'_0, M)$ are additional creep constants and $\dot{D}$ is given by (3.2). Now introduce S as the work-conjugate to $\dot{D}$:

$$S = \frac{\partial\omega}{\partial\dot{D}} = \sigma'_0\left(\frac{\dot{D}}{\dot{\varepsilon}'_0}\right)^M \tag{4.9}$$

and (4.8) thereby gives

$$s_\alpha^D = \frac{\partial \omega}{\partial \dot{d}_\alpha} = \frac{S}{\dot{D}} a_{\alpha\beta} \dot{d}_\beta \tag{4.10}$$

Substitution of (4.10) into (3.2) provides the direct relation between S and s_α^D as

$$S^2 = b_{\alpha\beta} s_\alpha^D s_\beta^D \tag{4.11}$$

where $b_{\alpha\beta} \equiv (a^{-1})_{\alpha\beta}$. The plastic work-rate at any point on Γ is non-negative:

$$s_\alpha^D \dot{d}_\alpha = S\dot{D} = \sigma_0' \dot{\varepsilon}_0' \left(\frac{\dot{D}}{\dot{\varepsilon}_0'}\right)^{M+1} \geqslant 0 \tag{4.12}$$

The constitutive dual $\omega^*(S)$ of $\omega(\dot{D})$ reads

$$\omega^*(S) = \sup_{\dot{D}} \left(S\dot{D} - \omega\left(\dot{D}\right)\right) = \frac{M}{M+1} \sigma_0' \dot{\varepsilon}_0' \left(\frac{S}{\sigma_0'}\right)^{(M+1)/M} \tag{4.13}$$

and the generalized strain rate $\dot{d}_\alpha$ is related to s_α according to

$$\dot{d}_\alpha = \frac{\partial \omega^*}{\partial s_\alpha} = \frac{\dot{D}}{S} b_{\alpha\beta} s_\beta^D \tag{4.14}$$

The above formulation is a generalization of Fredriksson and Gudmundson (2007) whose potential ω^* is reproduced by the choice $b_{12} = b_{21} = 0$ and $b_{11} = 4b_{22}$.

4.3. Minimum principle for the rate-dependent constitutive law

The above rate-dependent problem can be formulated via the successive use of two minimum principles.

4.3.1. Minimum Principle I and its dual

Assume that the displacement field $u_i(\boldsymbol{x})$ and the plastic strain $\varepsilon_{ij}^{PL}(\boldsymbol{x})$ are known everywhere in the current state, and the Cauchy stress $\sigma_{ij} = \sigma_{ij}^E$ is given by the elasticity relation (2.7). The energetic stresses ($\boldsymbol{r}^E$, $\boldsymbol{s}^E$) are known from (2.5) and (2.8). Suppose that the boundary of the solid is subjected to prescribed traction t_{ij}^0 over a portion S_T and to a prescribed plastic strain rate ($\dot{\varepsilon}_{ij}^{PL0}$) over the complementary portion S_U. Consider any kinematically admissible trial fields ($\dot{\boldsymbol{e}}^*, \dot{\boldsymbol{d}}^*$), and note that ($\boldsymbol{r}^{D*}$, $\boldsymbol{s}^{D*}$) follow immediately from (4.3) and (4.10). Then, the actual solution for ($\dot{\boldsymbol{e}}, \dot{\boldsymbol{d}}; \boldsymbol{r}, \boldsymbol{s}$) satisfies the following minimum statement,

$$H = \inf_{\dot{\varepsilon}_{ij}^{PL*}} \int_V \{\phi(\dot{E}^{P*}) + r_I^E \dot{e}_I^* - \sigma_{ij}' \dot{\varepsilon}_{ij}^{PL*}\}\, dV + \int_\Gamma \{\omega(\dot{D}^*) + s_\alpha^E \dot{d}_\alpha^*\}\, d\Gamma - \int_{S_T} \{t_{ij}^0 \dot{\varepsilon}_{ij}^{PL*}\}\, dS \tag{4.15}$$

Proof. Setting to zero the first variation of the r.h.s. of (4.15) with respect to $\dot{\varepsilon}_{ij}^{PL*}$ delivers the equilibrium relation (2.2ii) and the traction relations (2.3) and (2.4ii), along with the constitutive statements (4.3) and (4.10). Since $\phi(\dot{E}^{P*})$ and $\omega(\dot{D}^*)$ are strictly convex in their arguments, the functional possesses a unique minimum, and so the solution for $\dot{\varepsilon}_{ij}^{PL}$ is unique. The above proof is similar to that outlined by Hill (1956a) for a rigid, visco-plastic solid.

A dual principle also exists, as follows. Consider trial equilibrium fields ($\boldsymbol{r}^*$, $\boldsymbol{s}^*$). Then, with the actual solution written as ($\dot{\boldsymbol{e}}, \dot{\boldsymbol{d}}; \boldsymbol{r}, \boldsymbol{s}$) we find that

$$\begin{aligned} I &= \int_{S_U} \{n_k \tau_{ijk} \dot{\varepsilon}_{ij}^{PL0}\}\, dS - \int_V \{\phi^*(\Sigma)\}\, dV - \int_\Gamma \{\omega^*(S)\}\, d\Gamma \\ &= \sup_{(\boldsymbol{r}^*, \boldsymbol{s}^*)} \int_{S_U} \{n_k \tau_{ijk}^* \dot{\varepsilon}_{ij}^{PL0}\}\, dS - \int_V \{\phi^*(\Sigma^*)\}\, dV - \int_\Gamma \{\omega^*(S^*)\}\, d\Gamma \end{aligned} \tag{4.16}$$

with equality attained when ($\boldsymbol{r}^*$, $\boldsymbol{s}^*$) is the actual solution. The proof follows immediately from the principle of virtual work (2.1), together with the strict convexity of $\phi^*(\Sigma)$ and of $\omega^*(S)$. □

4.4. Minimum Principle II and its dual

Assume that $\dot{\varepsilon}_{ij}^{PL}$ is known upon making use of the above Minimum Principle I. Then, the velocity field $\dot{u}_i$ is obtained from a principle similar to the classical minimum principle for an elastic solid. By minimizing the functional:

$$J(\dot{u}_i^*) = \int_V \left\{\frac{1}{2} L_{ijkl} (\dot{\varepsilon}_{ij}^* - \dot{\varepsilon}_{ij}^{PL})(\dot{\varepsilon}_{kl}^* - \dot{\varepsilon}_{kl}^{PL})\right\} dV - \int_{S_T} \{\dot{T}_i^0 \dot{u}_i^*\}\, dS \tag{4.17}$$

over all $\dot{u}_i^*$, such that $\dot{u}_i^* = \dot{u}_i^0$ on S_U, we obtain the unique actual solution $\dot{u}_i$, along with the equilibrium relation (2.2i) in rate form. The stress rate $\dot{\sigma}_{ij}$ follows immediately using the rate form of (2.7).

An alternative dual formulation can be used to obtain $\dot{\sigma}_{ij}$ uniquely. Write $M_{ijkl}\equiv(L^{-1})_{ijkl}$ as the elastic compliance, and introduce the functional

$$F(\dot{\sigma}^*_{ij}) \equiv \int_V \left\{ \frac{1}{2} M_{ijkl}\dot{\sigma}^*_{ij}\dot{\sigma}^*_{kl} + \dot{\sigma}^*_{ij}\dot{\varepsilon}^{PL}_{ij} \right\} dV - \int_{S_U} \{\dot{T}^*_i \dot{u}^0_i\}\, dS \tag{4.18}$$

Then, $F(\dot{\sigma}^*_{ij})$ is minimized over all trial equilibrium fields with $\dot{\sigma}^*_{ij} n_j = \dot{T}^0_i$ on S_T in order to deliver the actual solution. The proof is again standard.

5. Rate-independent version

An elasto-plastic, rate-independent version can be derived by taking the limit $N\to 0$ and $M\to 0$. Strain hardening is admitted explicitly by replacing σ_0 by $\Sigma_Y(E^P)$ and σ'_0 by $S_Y(D)$, where $E^P = \int \dot{E}^P\, dt$ and $D = \int \dot{D}\, dt$.

5.1. Yield surface and hardening rule in V

Upon taking the rate-independent limit, the flow rule (4.7) reduces to the normality rule

$$\dot{e}_I = \frac{\partial \phi^*}{\partial r^D_I} = \dot{E}^P \frac{\partial \Sigma}{\partial r^D_I}, \quad \Sigma(\boldsymbol{r}^D) = \Sigma_Y(E^P) \tag{5.1}$$

provided $\dot{E}^P > 0$. Thus, the plastic strain rate vector $\dot{\boldsymbol{e}}$ lies along the outward normal to the yield surface $\Sigma(\boldsymbol{r}^D) = \Sigma_Y(E^P)$. When $\Sigma(\boldsymbol{r}^D)$ is less than $\Sigma_Y(E^P)$ the response is elastic and $\dot{E}^P = \dot{e}_I = 0$; then, $\boldsymbol{r}^D$ is indeterminate, but must still satisfy the balance law (2.2ii). Alternatively, $\dot{E}^P$ is non-vanishing and $\boldsymbol{r}^D$ is given in terms of $\dot{\boldsymbol{e}}$ via (4.3), with $\Sigma = \Sigma_Y(E^P)$. We further note that the yield surface is shifted from the origin by the energetic stress $\boldsymbol{r}^E$. The fact that $\boldsymbol{r}^D$ is indeterminate in the elastic zone has a major effect upon the mathematical structure of the incremental plasticity law. For example, it is not possible to use the relation $\Sigma(\boldsymbol{r}^D) \leqslant \Sigma_Y(E^P)$ in order to check on the attainment of yield. This is analogous to the case of a conventional, rigid-plastic solid where the Cauchy stress is not defined within a rigid region, and cannot be used in order to test for the onset of yield. Hill (1951, 1956b) gave the solution to the paradox for the rigid-plastic solid by outlining the variational structure and associated uniqueness and extremum principles. We follow a similar strategy here for the strain-gradient solid.

The hardening rule follows directly from the consistency relation $\dot{\Sigma} = h(E^P)\dot{E}^P$ where $h \equiv d\Sigma_Y/dE^P$. Consequently, (5.1) can be written as

$$\dot{\boldsymbol{e}} = \frac{1}{h}\frac{\partial \Sigma}{\partial \boldsymbol{r}^D}\left(\frac{\partial \Sigma}{\partial \boldsymbol{r}^D}\cdot \dot{\boldsymbol{r}}^D\right) \tag{5.2}$$

Upon making use of the identity $\dot{r}^D_I = \dot{r}_I - \dot{r}^E_I$ we can re-write (5.2) in the usual form for associated plastic flow with kinematic hardening,

$$\dot{\boldsymbol{e}} = \frac{1}{H}\frac{\partial \Sigma}{\partial \boldsymbol{r}^D}\left(\frac{\partial \Sigma}{\partial \boldsymbol{r}^D}\cdot \dot{\boldsymbol{r}}\right) \tag{5.3}$$

with H determined as follows. First, note that $(\dot{Q}^E_{ij}, \dot{\tau}^E_{ijk})$ follows directly from (2.5) as

$$\dot{Q}^E_{ij} = \frac{\partial^2 U}{\partial \varepsilon^{PL}_{ij}\partial \varepsilon^{PL}_{kl}}\dot{\varepsilon}^{PL}_{kl} + \frac{\partial^2 U}{\partial \varepsilon^{PL}_{ij}\partial \varepsilon^{PL}_{kl,m}}\dot{\varepsilon}^{PL}_{kl,m} \tag{5.4}$$

and

$$\dot{\tau}^E_{ijk} = \frac{\partial^2 U}{\partial \varepsilon^{PL}_{ij,k}\partial \varepsilon^{PL}_{mn}}\dot{\varepsilon}^{PL}_{mn} + \frac{\partial^2 U}{\partial \varepsilon^{PL}_{ij,k}\partial \varepsilon^{PL}_{lm,n}}\dot{\varepsilon}^{PL}_{lm,n} \tag{5.5}$$

and these linear relations can be written symbolically as

$$\dot{r}^E_I = B_{IJ}\dot{e}_J \tag{5.6}$$

in terms of the 20×20 matrix of coefficients B_{IJ}. This expression gives the evolution law for the centre of the yield surface. Substitution of (5.6) into $\dot{r}^D_I = \dot{r}_I - \dot{r}^E_I$ and suitable rearrangement of (5.2) leads to the result (5.3), with

$$H \equiv h + \frac{1}{\Sigma^2} r^D_P D_{PJ} B_{JK} D_{KQ} r^D_Q \tag{5.7}$$

Given $\dot{\boldsymbol{e}}$, the component of $\dot{\boldsymbol{r}}^D$ along the normal to the yield surface is known from the constitutive law, *but not* its tangential component. But given $\dot{\boldsymbol{e}}$ and $\dot{\varepsilon}_{ij}$, the stress rate $\dot{\sigma}_{ij} = \dot{\sigma}^E_{ij}$ is known from the rate form of the usual elasticity relation (2.7).

It should perhaps be noted that, while the present structure provides kinematic hardening, this does not preclude the possibility of constructing an alternative theory in which the centre of the yield surface is described wholly or partly by a dissipative stress.

5.2. Yield surface and hardening rule on Γ

In the rate-independent limit, the flow rule (4.14) reduces to the normality statement

$$\dot{d}_\alpha = \frac{\partial \omega^*}{\partial s^D_\alpha} = \dot{D}\frac{\partial S}{\partial s^D_\alpha}, \quad S(\boldsymbol{s}^D) = S_Y(D) \tag{5.8}$$

provided $\dot{D} > 0$. Thus, the plastic strain rate vector $\dot{\boldsymbol{d}}$ lies along the outward normal to the yield surface $S(\boldsymbol{s}^D) = S_Y(D)$. When $S(\boldsymbol{s}^D)$ is less than $S_Y(D)$ the response is elastic and $\dot{D} = \dot{d}_\alpha = 0$: $\boldsymbol{s}^D$ is indeterminate, but must still satisfy the balance law (2.3). Dually, for non-vanishing $\dot{D}$, $\boldsymbol{s}^D$ is given in terms of $\dot{\boldsymbol{d}}$ via (4.10), with $S = S_Y(D)$.

Plastic loading implies the consistency relation $\dot{S} = g(D)\dot{D}$ where $g \equiv dS_Y/dD$. Consequently, (5.8) can be written in the usual form for associated plastic flow,

$$\dot{\boldsymbol{d}} = \frac{1}{g}\frac{\partial S}{\partial \boldsymbol{s}^D}\left(\frac{\partial S}{\partial \boldsymbol{s}^D}\cdot \dot{\boldsymbol{s}}^D\right) \tag{5.9}$$

Now make use of the identity $\dot{s}^D_I = \dot{s}_I - \dot{s}^E_I$ to re-write (5.9) as

$$\dot{\boldsymbol{d}} = \frac{1}{G}\frac{\partial S}{\partial \boldsymbol{s}^D}\left(\frac{\partial S}{\partial \boldsymbol{s}^D}\cdot \dot{\boldsymbol{s}}\right) \tag{5.10}$$

The hardening coefficient G is directly linked to g since

$$\dot{p}^E_{ij} = \frac{\partial^2 U^S}{\partial[\varepsilon^{PL}_{ij}]\partial[\varepsilon^{PL}_{kl}]}[\dot{\varepsilon}^{PL}_{kl}] + \frac{\partial^2 U^S}{\partial[\varepsilon^{PL}_{ij}]\partial\langle\varepsilon^{PL}_{kl}\rangle}\langle\dot{\varepsilon}^{PL}_{kl}\rangle \tag{5.11}$$

and

$$\dot{q}^E_{ij} = \frac{\partial^2 U^S}{\partial\langle\varepsilon^{PL}_{ij}\rangle\partial[\varepsilon^{PL}_{kl}]}[\dot{\varepsilon}^{PL}_{kl}] + \frac{\partial^2 U^S}{\partial\langle\varepsilon^{PL}_{ij}\rangle\partial\langle\varepsilon^{PL}_{kl}\rangle}\langle\dot{\varepsilon}^{PL}_{kl}\rangle \tag{5.12}$$

and these linear relations can be written symbolically as

$$\dot{s}^E_\alpha = C_{\alpha\beta}\dot{d}_\beta \tag{5.13}$$

Substitution of (5.13) into $\dot{s}^D_I = \dot{s}_I - \dot{s}^E_I$ and suitable rearrangement of (5.9) leads to the result (5.10), with

$$G \equiv g + \frac{1}{S^2}s^D_\alpha b_{\beta\chi}C_{\chi\eta}b_{\eta\lambda}s^D_\lambda \tag{5.14}$$

In the remainder of this paper we shall focus on the rate-independent case. A unique solution to a boundary problem at any instant requires knowledge of both the current surface traction and its rate. The solution process follows the same sequence as was outlined in Part I and involves the application of two minimum principles. These minimum principles are variants of the corresponding principles presented in Part I, now admitting variations with respect to the tensor $\dot{\varepsilon}^{PL}_{ij}$ rather than the scalar $\dot{\varepsilon}^P$. The mathematical structure of these principles (and the attendant uniqueness statements) have close parallels with the development of Hill (1951, 1956b) for a conventional rigid-hardening solid.

6. Minimum Principle I to obtain the stress quantities (*r*, *s*)

Suppose that the boundary of the elasto-plastic solid is subjected to prescribed traction fields $(T^0_i,\ t^0_{ij})$ over a portion S_T and to the displacement-type loading $(u^0_i,\ \varepsilon^{PL0}_{ij})$ over the complementary portion S_U. Assume that the stress $\sigma_{ij}(\boldsymbol{x})$ and the plastic strain ε^{PL}_{ij} are known everywhere. Write $(\dot{\boldsymbol{e}}, \dot{\boldsymbol{d}}; \boldsymbol{r}, \boldsymbol{s})$ as the actual solution, and recall that $(\boldsymbol{r}, \boldsymbol{s})$ is given by (4.3) and (4.10) in the active plastic zone and is indeterminate in the elastic zone. Consider any kinematically admissible trial fields $(\dot{\boldsymbol{e}}^*, \dot{\boldsymbol{d}}^*)$, and note that $\boldsymbol{r}^*$ follows immediately from (4.3) provided $\dot{\boldsymbol{e}}^* \neq 0$. If $\dot{\boldsymbol{e}}^* = 0$ then $\boldsymbol{r}^*$ is indeterminate but is within or on the yield surface. Likewise, $\boldsymbol{s}^*$ follows immediately from (4.10) provided $\dot{\boldsymbol{d}}^* \neq 0$. If $\dot{\boldsymbol{d}}^* = 0$ then $\boldsymbol{s}^*$ is indeterminate but is within or on the yield surface. Then, the actual solution for $(\dot{\boldsymbol{e}}, \dot{\boldsymbol{d}}; \boldsymbol{r}, \boldsymbol{s})$ satisfies the following minimum statement:

$$\begin{aligned} H &= \int_{S_U}\{t_{ij}\dot{\varepsilon}^{PL0}_{ij}\}\, dS \\ &= \inf_{\dot{\varepsilon}^{PL*}_{ij}} \int_V \{\Sigma_Y \dot{E}^{P*} + r^E_I \dot{e}^*_I - \sigma'_{ij}\dot{\varepsilon}^{PL*}_{ij}\}\, dV + \int_\Gamma \{S_Y \dot{D}^* + s^E_\alpha \dot{d}^*_\alpha\}\, d\Gamma - \int_{S_T}\{t^0_{ij}\dot{\varepsilon}^{PL*}_{ij}\}\, dS \end{aligned} \tag{6.1}$$

Proof. The relations (5.1) and (5.8) state that $(\dot{\boldsymbol{e}}, \dot{\boldsymbol{d}})$ are normal to their respective yield surfaces. Also, the yield surfaces $\Sigma(\boldsymbol{r}) = \Sigma_Y$ and $S(\boldsymbol{s}) = S_Y$ are convex, and so the 'maximum plastic work principle for a material element' reads

$$\text{in } V: \quad (r_I^* - r_I)\dot{e}_I^* \geqslant 0 \text{ and } (r_I - r_I^*)\dot{e}_I \geqslant 0 \tag{6.2}$$

$$\text{on } \Gamma: \quad (s_\alpha^* - s_\alpha)\dot{d}_\alpha^* \geqslant 0 \text{ and } (s_\alpha - s_\alpha^*)\dot{d}_\alpha \geqslant 0 \tag{6.3}$$

Now make use of the balance laws (2.2ii) and (2.3) to obtain

$$\begin{aligned}\int_V \{r_I^* \dot{e}_I^*\}\, dV + \int_\Gamma \{s_\alpha^* \dot{d}_\alpha^*\}\, d\Gamma &\geqslant \int_V \{r_I \dot{e}_I^*\}\, dV + \int_\Gamma \{s_\alpha \dot{d}_\alpha^*\}\, d\Gamma \\ &= \int_V \{\sigma'_{ij} \dot{\varepsilon}_{ij}^{PL*}\}\, dV + \int_S \{n_k \tau_{ijk} \dot{\varepsilon}_{ij}^{PL*}\}\, dS\end{aligned} \tag{6.4}$$

Upon recalling that $r_I^* \dot{e}_I^* = \Sigma_Y \dot{E}^{P*} + r_I^E \dot{e}_I^*$ and that $s_\alpha^* \dot{d}_\alpha^* = S_Y \dot{D}^* + s_\alpha^E \dot{d}_\alpha^*$ the above inequality can be re-arranged to the form of (6.1). This Minimum Principle delivers unique values for $(\boldsymbol{r}, \boldsymbol{s})$ in the active plastic zone (see Appendix A for a discussion of uniqueness). The plastic strain rates $(\dot{\boldsymbol{e}}, \dot{\boldsymbol{d}})$ are also unique up to a multiplicative factor $\dot{\Lambda}$ which is constant in any active plastic zone since the functional in (6.1) is homogeneous and of degree one in $\dot{\varepsilon}_{ij}^{PL*}$. In order to define $\dot{\Lambda}$ unambiguously, introduce a strain rate field of 'unit magnitude' $\hat{\varepsilon}_{ij}(\boldsymbol{x})$ within each active plastic zone V_a, such that

$$\frac{1}{V_a}\int_{V_a} \{\hat{\varepsilon}_{ij}\hat{\varepsilon}_{ij}\}\, dV_a = 1 \tag{6.5}$$

Then, write $\dot{\Lambda}$ as the scaling factor for the plastic strain rate such that

$$\dot{\varepsilon}_{ij}^{PL}(\boldsymbol{x}) = \dot{\Lambda}\hat{\varepsilon}_{ij}(\boldsymbol{x}) \quad \square \tag{6.6}$$

6.1. *Dual formulation*

Consider trial equilibrium fields $(\boldsymbol{r}^*, \boldsymbol{s}^*)$ that do not violate the yield condition at any point. Then, with the actual solution written as $(\dot{\boldsymbol{e}}, \dot{\boldsymbol{d}}; \boldsymbol{r}, \boldsymbol{s})$ we find that

$$I = \int_{S_U} \{n_i \tau_{ijk} \dot{\varepsilon}_{jk}^{PL0}\}\, dS = \sup_{(\boldsymbol{r}^*, \boldsymbol{s}^*)} \int_{S_U} \{n_k \tau_{ijk}^* \dot{\varepsilon}_{ij}^{PL0}\}\, dS \tag{6.7}$$

with equality attained when $(\boldsymbol{r}^*, \boldsymbol{s}^*)$ is the actual solution. The proof follows immediately from the principle of virtual work (2.1), together with (6.2) and (6.3). Relation (6.7) provides a lower bound restriction on t_{ij} while (6.1) provides an upper bound, in direct correspondence to the classical theorems of limit load analysis Hill (1948, 1951).

7. Minimum Principle II to obtain the plastic multipliers and the velocity field

Assume that the stress $\sigma_{ij}(\boldsymbol{x})$ is known everywhere, and $(\boldsymbol{r}, \boldsymbol{s})$ is known in the active plastic zone upon making use of Minimum Principle I. Write $\dot{\varepsilon}_{ij}^{PL}(\boldsymbol{x})$ in terms of the plastic multiplier $\dot{\Lambda}$ and the known 'unit' field $\hat{\varepsilon}_{ij}(\boldsymbol{x})$ as given by relation (6.5). Recall that $\dot{\Lambda}$ is constant within each active plastic zone, but is as yet undetermined. Then, minimizing the functional

$$\begin{aligned}J(\dot{u}_i^*, \dot{\Lambda}^*) = &\frac{1}{2}\int_V \{L_{ijkl}(\dot{\varepsilon}_{ij}^* - \dot{\varepsilon}_{ij}^{PL*})(\dot{\varepsilon}_{kl}^* - \dot{\varepsilon}_{kl}^{PL*}) + H\dot{E}^{P*2}\}\, dV \\ &+ \frac{1}{2}\int_\Gamma \{G\dot{D}^{*2}\}\, d\Gamma - \int_{S_T} \{\dot{T}_i^0 \dot{u}_i^* + \dot{t}_{ij}^0 \dot{\varepsilon}_{ij}^{P*}\}\, dS\end{aligned} \tag{7.1}$$

over all $(\dot{u}_i^*, \dot{\Lambda}^* \geqslant 0)$ delivers the actual solution $(\dot{u}_i, \dot{\Lambda})$. In (7.1), $\dot{\varepsilon}_{ij}^{PL*} = \dot{\Lambda}^* \hat{\varepsilon}_{ij}$.

Proof. Consider the trial solution $(\dot{u}_i^*, \dot{E}^{P*}, \dot{D}^*)$ in addition to the actual solution $(\dot{u}_i, \dot{E}^P, \dot{D})$ and write $\Delta\dot{u}_i \equiv \dot{u}_i^* - \dot{u}_i$, $\Delta\dot{E}^P \equiv \dot{E}^{P*} - \dot{E}^P$, $\Delta\dot{D} \equiv \dot{D}^* - \dot{D}$ and $\Delta J \equiv J(\dot{u}_i^*, \dot{\Lambda}^*) - J(\dot{u}_i, \dot{\Lambda})$. It follows that

$$\begin{aligned}\Delta J \geqslant &\int_V \{L_{ijkl}\dot{\varepsilon}_{kl}^{EL}\Delta\dot{\varepsilon}_{ij}^{EL} + HA_{IJ}\dot{e}_J\Delta\dot{e}_I\} dV \\ &+ \int_\Gamma \{Ga_{\alpha\beta}\dot{d}_\beta\Delta\dot{d}_\alpha\} d\Gamma - \int_{S_T} \{\dot{T}_i^0 \Delta\dot{u}_i + \dot{t}_{ij}^0 \Delta\dot{\varepsilon}_{ij}^{PL}\} dS\end{aligned} \tag{7.2}$$

and this may be re-written as

$$\Delta J \geqslant \int_V \{(HA_{IJ}\dot{e}_J - \dot{r}_I)\Delta\dot{e}_I\}\, dV + \int_\Gamma \{(Ga_{\alpha\beta}\dot{d}_\beta - \dot{s}_\alpha)\Delta\dot{d}_\alpha\}\, d\Gamma \tag{7.3}$$

via the principle of virtual work (2.1). We can demonstrate that the right-hand side of (7.3) is non-negative by proving that $\dot{r}_I\, \Delta\dot{e}_I \leqslant HA_{IJ}\dot{e}_J\, \Delta\dot{e}_I$ and $\dot{s}_\alpha\, \Delta\dot{d}_\alpha \leqslant Ga_{\alpha\beta}\dot{d}_\beta\, \Delta\dot{d}_\alpha$ at each point. The detailed argument is omitted here, as it routinely follows that of Hill (1956b), and of Part I. In brief, we show that $\dot{r}_I\,\Delta\dot{e}_I \leqslant HA_{IJ}\dot{e}_J\,\Delta\dot{e}_I$ by considering in turn the four possibilities: (i) $\dot{E}^P > 0$, $\dot{E}^{P*} > 0$; (ii) $\dot{E}^P > 0$, $\dot{E}^{P*} = 0$; (iii) $\dot{E}^P = 0$, $\dot{E}^{P*} > 0$; (iv) $\dot{E}^P = 0$, $\dot{E}^{P*} = 0$. Likewise, to show that $\dot{s}_\alpha\,\Delta\dot{d}_\alpha \leqslant Ga_{\alpha\beta}\dot{d}_\beta\,\Delta\dot{d}_\alpha$, we consider the analogous four possibilities. Uniqueness of the solution is demonstrated in Appendix A. □

Further manipulation of (7.1) using (5.3) and (5.10) allows ΔJ in (7.3) to be re-expressed as

$$\Delta J \geqslant \int_V \{(H\dot{E}^P - \dot{\Sigma})\Delta\dot{E}^P\}\, dV + \int_\Gamma \{(G\dot{D} - \dot{S})\Delta\dot{D}\}\, d\Gamma \tag{7.4}$$

Straightforward manipulations can be used to show that at the solution,

$$2J_{min} = \int_{S_U} \{\dot{T}_i\dot{u}_i^0 + \dot{t}_{ij}\dot{\varepsilon}_{ij}^{PL0}\}\, dS - \int_{S_T} \{\dot{T}_i^0\dot{u}_i + \dot{t}_{ij}^0\dot{\varepsilon}_{ij}^{PL}\}\, dS \tag{7.5}$$

7.1. Dual formulation

An alternative dual formulation can be used to obtain $(\dot{\sigma}_{ij}, \dot{\Sigma}, \dot{S})$ uniquely. Write $M_{ijkl} \equiv (L^{-1})_{ijkl}$ as the elastic compliance, and introduce the functional

$$F(\dot{\sigma}_{ij}^*, \dot{r}_I^*, \dot{s}_\alpha^*) \equiv \frac{1}{2}\int_V \left\{ M_{ijkl}\dot{\sigma}_{ij}^*\dot{\sigma}_{kl}^* + \frac{1}{H}\left(\frac{D_{IJ}r_J^D\dot{r}_I^*}{\Sigma}\right)^2 \right\} dV + \int_\Gamma \left\{ \frac{1}{2G}\left(\frac{b_{\alpha\beta}s_\beta^D\dot{s}_\alpha^*}{S}\right)^2 \right\} d\Gamma - \int_{S_U} \{\dot{T}_i^*\dot{u}_i^0 + \dot{t}_{ij}^*\dot{\varepsilon}_{ij}^{PL0}\}\, dS \tag{7.6}$$

Then, $F(\dot{\sigma}_{ij}^*, \dot{r}_I^*, \dot{s}_\alpha^*)$ is minimized over all equilibrium fields in order to deliver the actual solution.

The proof is based upon the principle of virtual work (2.1) written in the form

$$\int_S \{\Delta\dot{T}_i\dot{u}_i + \Delta\dot{t}_{ij}\dot{\varepsilon}_{ij}^{PL}\}\, dS = \int_V \{\Delta\dot{\sigma}_{ij}\dot{\varepsilon}_{ij}^{EL} + \Delta\dot{r}_I\dot{e}_I\}\, dV + \int_\Gamma \{\Delta\dot{s}_\alpha\dot{d}_\alpha\}\, d\Gamma \tag{7.7}$$

where the symbol Δ again denotes the difference between the trial equilibrium field solution (asterisked) and the actual solution (no asterisk). Substitute the following inequalities into (7.7):

$$\Delta\dot{\sigma}_{ij}\dot{\varepsilon}_{ij}^{EL} \leqslant \frac{1}{2}M_{ijkl}\dot{\sigma}_{ij}^*\dot{\sigma}_{kl}^* - \frac{1}{2}M_{ijkl}\dot{\sigma}_{ij}\dot{\sigma}_{kl} \tag{7.8}$$

$$\text{for } \dot{E}^P > 0: \quad \Delta\dot{r}_I\dot{e}_I \leqslant \frac{1}{2H}\left(\frac{D_{IJ}r_J^D\dot{r}_I^*}{\Sigma}\right)^2 - \frac{1}{2H}\left(\frac{D_{IJ}r_J^D\dot{r}_I}{\Sigma}\right)^2 \tag{7.9}$$

$$\text{and for } \dot{D} > 0: \quad \Delta\dot{s}_\alpha\dot{d}_\alpha \leqslant \frac{1}{2G}\left(\frac{b_{\alpha\beta}s_\beta^D\dot{s}_\alpha^*}{S}\right)^2 - \frac{1}{2G}\left(\frac{b_{\alpha\beta}s_\beta^D\dot{s}_\alpha}{S}\right)^2 \tag{7.10}$$

to obtain

$$F(\dot{\sigma}_{ij}, \dot{r}_I, \dot{s}_\alpha) \leqslant F(\dot{\sigma}_{ij}^*, \dot{r}_I^*, \dot{s}_\alpha^*) \tag{7.11}$$

Finally, we note that $F(\dot{\sigma}_{ij}, \dot{r}_I, \dot{s}_\alpha)$ can be written in the form

$$-2F(\dot{\sigma}_{ij}, \dot{r}_I, \dot{s}_\alpha) = \int_{S_U} \{\dot{T}_i\dot{u}_i^0 + \dot{t}_{ij}\dot{\varepsilon}_{ij}^{PL0}\}\, dS - \int_{S_T} \{\dot{T}_i^0\dot{u}_i + \dot{t}_{ij}^0\dot{\varepsilon}_{ij}^{PL}\}\, dS \tag{7.12}$$

and so the values of $2J$ in the extremum of (7.1) and of $-2F$ in the extremum of (7.12) approximate the quantity $\int_{S_U}\{\dot{T}_i\dot{u}_i^0 + \dot{t}_{ij}\dot{\varepsilon}_{ij}^{PL0}\}\, dS - \int_{S_T}\{\dot{T}_i^0\dot{u}_i + \dot{t}_{ij}^0\dot{\varepsilon}_{ij}^{PL}\}\, dS$ from above and below.

8. The rigid-hardening solid

The above rate-independent theory can be reduced to a rigid-hardening version absent elasticity.[1] This modified constitutive law is the natural extension to classical, rigid-hardening theory as expounded by Hill (1956b). Uniqueness and extremum principles persist and limit theorems follow immediately from them. In the absence of elastic straining, the incompressible plastic strain field is derived directly from the displacement field:

$$\dot{\varepsilon}_{ij}^{PL} \equiv \tfrac{1}{2}(\dot{u}_{i,j} + \dot{u}_{j,i}), \quad \dot{\varepsilon}_{kk}^{PL} = 0 \tag{8.1}$$

[1] This contrasts with the scalar version of Part I. A rigid-hardening version was not developed for the scalar version, for which only the effective stress σ_e is determined by the scalar analogue of (2.2ii).

The elastic term in the principle of virtual work (2.1) is now dropped and it becomes

$$\int_V \left\{ Q_{ij}\delta\varepsilon_{ij}^{PL} + \frac{1}{3}\sigma_{ii}\delta\varepsilon_{kk}^{PL} + \tau_{ijk}\delta\varepsilon_{ij,k}^{PL} \right\} dV + \int_\Gamma \{p_{ij}[\delta\varepsilon_{ij}^{PL}] + q_{ij}\langle\delta\varepsilon_{ij}^{PL}\rangle\}\, d\Gamma = \int_S \{T_i\delta u_i + t_{ij}\delta\varepsilon_{ij}^{PL}\}\, dS \tag{8.2}$$

along with the governing equilibrium relations (2.2)–(2.4). In (8.2), the hydrostatic component of Cauchy stress $\frac{1}{3}\sigma_{ii}$ is the Lagrange multiplier associated with the material incompressibility, $\dot{\varepsilon}_{kk}^{PL} = 0$. The deviatoric part of Cauchy stress is now defined by (2.2ii). The virtual work statement (8.2) can be simplified further upon exploiting (8.1). Introduce the notation ∂_i for the gradient operator, and on the surface S with unit normal n_j, decompose ∂_i into a normal operator $D \equiv n_i\partial_i$ and a surface gradient operator D_i such that

$$\partial_i = D_i + n_i D \tag{8.3}$$

Upon making use of Stokes' divergence theorem (see for example, Mindlin (1964) or Fleck and Hutchinson (1997)) the work principle (8.2) reduces to

$$\int_V \left\{ Q_{ij}\delta\varepsilon_{ij}^{PL} + \frac{1}{3}\sigma_{ii}\delta\varepsilon_{kk}^{PL} + \tau_{ijk}\delta\varepsilon_{ij,k}^{PL} \right\} dV + \int_\Gamma \left\{p_{ij}[\delta\varepsilon_{ij}^{PL}] + q_{ij}\langle\delta\varepsilon_{ij}^{PL}\rangle\right\} d\Gamma = \int_S \{\bar{T}_i\delta u_i + R_i(D\delta u_i)\}\, dS \tag{8.4}$$

where

$$\bar{T}_i = T_i - D_j t_{ij} + t_{ij}n_j(D_p n_p) = \sigma_{ij}n_j - D_j(\tau_{ijk}n_k) + \tau_{ijk}n_j n_k(D_p n_p) \tag{8.5}$$

and

$$R_i = t_{ij}n_j = \tau_{ijk}n_j n_k \tag{8.6}$$

Incompressibility dictates a kinematic restriction on the normal component of $D\delta u_i$ at the surface of the body:

$$n_i D\delta u_i = -D_j\delta u_j \tag{8.7}$$

Consequently, the number of independent higher order boundary conditions is reduced from 3 to 2, and the virtual work statement is altered. After some manipulation of (8.4), and upon making use of (8.7) we obtain

$$\int_V \left\{ Q_{ij}\delta\varepsilon_{ij}^{PL} + \frac{1}{3}\sigma_{ii}\delta\varepsilon_{kk}^{PL} + \tau_{ijk}\delta\varepsilon_{ij,k}^{PL} \right\} dV + \int_\Gamma \{p_{ij}[\delta\varepsilon_{ij}^{PL}] + q_{ij}\langle\delta\varepsilon_{ij}^{PL}\rangle\}\, d\Gamma = \int_S \{\hat{T}_i\delta u_i + \hat{R}_i(D\delta u_i)\}\, dS \tag{8.8}$$

where the two independent higher order tractions $\hat{R}_i$ tangential to S are

$$\hat{R}_i = R_i - n_i R_j n_j = \tau_{ijk}n_j n_k - n_i\tau_{pjk}n_p n_j n_k \tag{8.9}$$

and

$$\hat{T}_i = \bar{T}_i - n_i R_j n_j(D_p n_p) = \sigma_{ij}n_j - D_j(\tau_{ijk}n_k) + (\tau_{ijk}n_j n_k - n_i\tau_{pjk}n_p n_j n_k)(D_p n_p) \tag{8.10}$$

The yield surface and the flow laws for the rigid-hardening solid remain as before, as expressed by the relations (5.1)–(5.14).

Minimum principles (and the attendant uniqueness principles) can be developed for the rigid-hardening solid in a similar manner to that described above for the elastic–plastic solid. These principles along with their proofs are detailed in Appendix B.

9. Concluding remarks

The constitutive framework elaborated above for both rate-dependent and rate-independent solids can be implemented numerically using standard finite element methods and minimization procedures. This has been illustrated by Idiart et al. (2009) for the prototypical problem of a thin foil in bending.

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Appendix A. Uniqueness principles

A.1. Uniqueness Principle I for the stress quantities ($\boldsymbol{r}$, $\boldsymbol{s}$)

In the current state, the stress $\sigma_{ij}(\boldsymbol{x})$, displacement $u_i(\boldsymbol{x})$ and plastic strain $\varepsilon_{ij}^{PL}(\boldsymbol{x})$ are known everywhere. We shall now show that the generalized stress ($\boldsymbol{r}$, $\boldsymbol{s}$) is unique within each active plastic zone, while the plastic strain rate $\dot{\varepsilon}_{ij}^{PL}(\boldsymbol{x})$ is known uniquely up to an arbitrary plastic multiplier $\dot{\Lambda}$, as defined by (6.6). For the case where $\dot{\varepsilon}_{ij}^{PL}(\boldsymbol{x})$ is specified at some point, the associated multiplier $\dot{\Lambda}$ is determined.

Consider two solutions for ($\boldsymbol{r}$, $\boldsymbol{s}$) and for ($\dot{u}_i$, $\dot{\varepsilon}^P$), and denote one solution by an asterisk and the other without an asterisk. The difference between the two solutions is denoted by the symbol Δ. Then, the principle of virtual work (2.1) gives

$$0 = \int_S \{\Delta T_i \Delta \dot{u}_i + \Delta t_{ij} \Delta \dot{\varepsilon}_{ij}^{PL}\}\, dS = \int_V \{\Delta \sigma_{ij} \Delta \dot{\varepsilon}_{ij}^{EL} + \Delta r_I \Delta \dot{e}_I\}\, dV + \int_\Gamma \{\Delta s_\alpha \Delta \dot{d}_\alpha\}\, d\Gamma \tag{A.1}$$

We assert that $\Delta\sigma_{ij} = 0$. Now, (6.2) and (6.3) give $\Delta r_I \Delta \dot{e}_I \geqslant 0$ unless ($r_I{}^* = r_I$) and $\Delta s_\alpha \Delta \dot{d}_\alpha \geqslant 0$ unless ($s_\alpha{}^* = s_\alpha$). (A.1) implies that ($\boldsymbol{r}$, $\boldsymbol{s}$) are unique in the active plastic zone while $\Delta \dot{e}_I$ and $\Delta \dot{d}_\alpha$ may be non-zero. Within any given plastic zone, $\dot{\varepsilon}_{ij}^{PL}(\boldsymbol{x})$ is known up to the value of the arbitrary plastic multiplier $\dot{\Lambda}$.

Further, the value of $\dot{\Lambda}$ is known for any active plastic zone when a non-zero $\dot{\varepsilon}_{ij}^{PL}(\boldsymbol{x})$ is prescribed on a portion of the boundary of the plastic zone. This can occur when a portion S_U of the surface of the body is subjected to prescribed non-zero $\dot{\varepsilon}_{ij}^{PL}(\boldsymbol{x})$.

A.2. Uniqueness Principle II for the plastic multiplier

Assume that the stress $\sigma_{ij}(\boldsymbol{x})$ is known everywhere, and ($\boldsymbol{r}$,$\boldsymbol{s}$) are known in the active plastic zone. We shall now show that a stipulation of the traction rate data is sufficient to ensure a unique solution for $\dot{\Lambda}$ within each active plastic zone and thereby $\dot{\varepsilon}_{ij}^{PL}(\boldsymbol{x})$.

Consider two solutions for ($\dot{\boldsymbol{r}}$, $\dot{\boldsymbol{s}}$) and for ($\dot{u}_i$, $\dot{\varepsilon}_{ij}^{PL}$), and denote the first solution by a superscript asterisk (*), and the second solution by the asterisk absent. The first solution minus the second solution is labeled by the symbol Δ. Then, the principle of virtual work (2.1) implies

$$0 = \int_S \{\Delta \dot{T}_i \Delta \dot{u}_i + \Delta \dot{t}_{ij} \Delta \dot{\varepsilon}_{ij}^{PL}\}\, dS = \int_V \{\Delta \dot{\sigma}_{ij} \Delta \dot{\varepsilon}_{ij}^{EL} + \Delta \dot{r}_I \Delta \dot{e}_I\}\, dV + \int_\Gamma \{\Delta \dot{s}_\alpha \Delta \dot{d}_\alpha\}\, d\Gamma \tag{A.2}$$

Consider the integrand on the right-hand side of (A.2). We shall show that it is positive unless the two solutions coincide. Assume that the elastic modulus L_{ijkl} is positive definite. Then,

$$\Delta \dot{\sigma}_{ij} \Delta \dot{\varepsilon}_{ij}^{EL} = L_{ijkl} \Delta \dot{\varepsilon}_{ij}^{EL} \Delta \dot{\varepsilon}_{kl}^{EL} \geqslant 0 \tag{A.3}$$

Second, note that (5.1) and (5.8) deliver

$$\Delta \dot{r}_I \Delta \dot{e}_I = \Delta \dot{\Sigma} \Delta \dot{E}^P \text{ and } \Delta \dot{s}_\alpha \Delta \dot{d}_\alpha = \Delta \dot{S} \Delta \dot{D} \tag{A.4}$$

Now, follow the argument of Hill (1956b), and of Part I, by considering in turn the four possibilities: (i) $\dot{E}^P > 0, \dot{E}^{P*} > 0$; (ii) $\dot{E}^P > 0, \dot{E}^{P*} = 0$; (iii) $\dot{E}^P = 0, \dot{E}^{P*} > 0$; and (iv) $\dot{E}^P = 0, \dot{E}^{P*} = 0$. These lead to the result that $\Delta\dot{\Sigma}\Delta\dot{E}^P \geqslant H(\Delta \dot{E}^P)^2 \geqslant 0$ provided $H > 0$, and likewise $\Delta \dot{S} \Delta \dot{D} \geqslant G(\Delta \dot{D})^2 \geqslant 0$, provided $G > 0$.

The work statement (A.2) implies that the elastic strain rate $\dot{\varepsilon}_{kl}^{EL}$ and the plastic multipliers ($\dot{E}^P$, $\dot{D}$) are unique. Consequently, $\dot{\sigma}_{ij}$, $\dot{\Lambda}$ and ($\dot{\boldsymbol{e}}$, $\dot{\boldsymbol{d}}$) are unique.

Appendix B. Minimum principles for the rigid-hardening solid

B.1. Minimum Principle I for the rigid-hardening solid

The Uniqueness Principle as given in Appendix A.1 also applies to the rigid-hardening solid. The principle reveals that the generalized stress ($\boldsymbol{r}$, $\boldsymbol{s}$) is unique within each active plastic zone, while the plastic strain rate $\dot{\varepsilon}_{ij}^{PL}(\boldsymbol{x})$ is known uniquely up to an arbitrary plastic multiplier $\dot{\Lambda}$, as defined by (6.6). For the case where a non-zero $\dot{\varepsilon}_{ij}^{PL}(\boldsymbol{x})$ is specified at some point, the associated multiplier $\dot{\Lambda}$ is determined. The Cauchy stress deviator σ'_{ij} is related to ($\boldsymbol{r}$, $\boldsymbol{s}$) via the balance law (2.2ii).

The minimum principle (6.1) simplifies considerably upon making use of the constraint (8.1) for the rigid-hardening solid. Assume that the stress $\sigma_{ij}(\boldsymbol{x})$ and the plastic strain ε_{ij}^{PL} are known everywhere. Write ($\dot{\boldsymbol{e}}, \dot{\boldsymbol{d}}; \boldsymbol{r}, \boldsymbol{s}$) as the actual solution, and recall that ($\boldsymbol{r}$, $\boldsymbol{s}$) is given by (4.3) and (4.10) in the active plastic zone and is indeterminate in the elastic zone. Consider any kinematically admissible trial fields ($\dot{\boldsymbol{e}}^*, \dot{\boldsymbol{d}}^*$), and note that $\boldsymbol{r}^*$ follows immediately from (4.3) provided $\dot{\boldsymbol{e}}^* \neq 0$. If $\dot{\boldsymbol{e}}^* = 0$ then $\boldsymbol{r}^*$ is indeterminate but is within or on the yield surface. Likewise, $\boldsymbol{s}^*$ follows immediately from (4.10) provided $\dot{\boldsymbol{d}}^* \neq 0$. If $\dot{\boldsymbol{d}}^* = 0$ then $\boldsymbol{s}^*$ is indeterminate but is within or on the yield surface. Then, the actual solution for ($\dot{\boldsymbol{e}}, \dot{\boldsymbol{d}}; \boldsymbol{r}, \boldsymbol{s}$) satisfies the following minimum statement:

$$H = \int_{S_U} \{\bar{T}_i \dot{u}_i^0 + R_i(D\dot{u}_i^0)\}\, dS$$
$$= \inf_{\dot{u}_i^*} \int_V \{\Sigma_Y \dot{E}^{P*} + r_I^E \dot{e}_I^*\}\, dV + \int_\Gamma \{S_Y \dot{D}^* + s_\alpha^E \dot{d}_\alpha^*\}\, d\Gamma - \int_{S_T} \{\bar{T}_i^0 \dot{u}_i^* + R_i^0(D\dot{u}_i^*)\}\, dS \tag{B.1}$$

The proof follows directly upon substitution of (6.2i) and (6.3i) into the principle of virtual work (8.4).

Upon admitting incompressibility, the principle of virtual work (8.8) can be used as the starting point, and a slightly modified version of the minimum principle will arise, such that the surface term in (B.1) now reads $\int_{S_T} \{\hat{T}_i^0 \dot{u}_i^* + \hat{R}_i^0(D\dot{u}_i^*)\}\, dS$. Similar modifications to the minimum principles stated below can be made in a straightforward manner (omitted for the sake of brevity).

B.1.1. Dual formulation

A dual formulation exists to (B.1). Instead of considering trial kinematic fields we now consider trial equilibrium fields ($\boldsymbol{r}^*$, $\boldsymbol{s}^*$) that do not violate the yield condition at any point. Write the actual solution as ($\dot{\boldsymbol{e}}, \dot{\boldsymbol{d}}; \boldsymbol{r}, \boldsymbol{s}$). Then, we find that

$$H = \int_{S_U} \{\bar{T}_i \dot{u}_i^0 + R_i(D\dot{u}_i^0)\}\, dS = \sup_{(\boldsymbol{r}^*,\boldsymbol{s}^*)} \int_{S_U} \{\bar{T}_i^* \dot{u}_i^0 + R_i^*(D\dot{u}_i^0)\}\, dS$$
$$= \sup_{(\boldsymbol{r}^*,\boldsymbol{s}^*)} \int_V \{Q_{ij}^* \dot{\varepsilon}_{ij}^{PL} + \tau_{ijk}^* \dot{\varepsilon}_{ij,k}^{PL}\}\, dV + \int_\Gamma \{s_\alpha^* \dot{d}_\alpha\}\, d\Gamma - \int_{S_T} \{\bar{T}_i^* \dot{u}_i + R_i^*(D\dot{u}_i)\}\, dS \tag{B.2}$$

with equality attained when ($\boldsymbol{r}^*$, $\boldsymbol{s}^*$) is the actual solution. This statement is analogous to Hill's (1948) maximum plastic work principle. The proof is a direct consequence of (6.2ii) and (6.3ii), along with the principle of virtual work (8.4). We conclude that (B.2) provides a lower bound restriction on H while (B.1) provides an upper bound, in direct correspondence to the classical theorems of limit load analysis. In fact, the trial fields from upper and lower bound analysis can be used immediately in order to obtain bounds on size effects. But now a word of caution: the augmented upper bound on collapse load may be infinite when the upper bound solution for the conventional solid has singular strain fields. A prototypical example is the circular fan in plane strain slip-line field analysis at the tip of a crack or beneath an indenter. In this case, the strain rate is of order $1/r$ and so the volume integral on the right-hand side of (B.1) is unbounded.

B.2. Minimum Principle II for the rigid-hardening solid

Assume that the stress $\sigma_{ij}(\boldsymbol{x})$ and ($\boldsymbol{r},\boldsymbol{s}$) are known in the active plastic zone upon making use of Minimum Principle I. Write $\dot{\varepsilon}_{ij}^{PL}(\boldsymbol{x})$ in terms of the plastic multiplier $\dot{\Lambda}$ and the known 'unit' field $\hat{\varepsilon}_{ij}(\boldsymbol{x})$ as given by relation (6.6). The field $\dot{\varepsilon}_{ij}^{PL}(\boldsymbol{x})$ is derivable from a continuous velocity field $\dot{u}_i(\boldsymbol{x})$ according to (8.1). We emphasise that $\dot{\Lambda}$ is constant within each active plastic zone, but is as yet undetermined. Consider the functional:

$$J(\dot{\Lambda}^*) = \frac{1}{2}\int_V \{H\dot{E}^{P*2}\}\, dV + \frac{1}{2}\int_\Gamma \{G\dot{D}^{*2}\}\, d\Gamma - \int_{S_T} \{\dot{\bar{T}}_i^0 \dot{u}_i^* + \dot{R}_i^0(D\dot{u}_i^*)\}\, dS \tag{B.3}$$

We shall show that this can be minimized over all $\dot{\Lambda}^* \geqslant 0$ to deliver the actual solution $\dot{\Lambda}$.

Proof. Consider the trial solution $(\dot{u}_i^*, \dot{E}^{P*}, \dot{D}^*)$ which scales with $\dot{\Lambda}^*$ in addition to the actual solution $(\dot{u}_i, \dot{E}^P, \dot{D})$ which scales with $\dot{\Lambda}$, and write $\Delta\dot{u}_i \equiv \dot{u}_i^* - \dot{u}_i$, $\Delta\dot{E}^P \equiv \dot{E}^{P*} - \dot{E}^P$, $\Delta\dot{D} \equiv \dot{D}^* - \dot{D}$ and $\Delta J \equiv J(\dot{\Lambda}^*) - J(\dot{\Lambda})$. It follows that

$$\Delta J \geqslant \int_V \{HA_{IJ}\dot{e}_J\Delta\dot{e}_I\}\, dV + \int_\Gamma \{Ga_{\alpha\beta}\dot{d}_\beta\Delta\dot{d}_\alpha\}\, d\Gamma - \int_{S_T} \{\dot{T}_i^0 \Delta\dot{u}_i + \dot{t}_{ij}^0 \Delta\dot{\varepsilon}_{ij}^{PL}\}\, dS \tag{B.4}$$

and, upon making use of the principle of virtual work (8.4) in rate form, this reduces to

$$\Delta J \geqslant \int_V \{(HA_{IJ}\dot{e}_J - \dot{r}_I)\Delta\dot{e}_I\}\, dV + \int_\Gamma \{(Ga_{\alpha\beta}\dot{d}_\beta - \dot{s}_\alpha)\,\Delta\dot{d}_\alpha\}\, d\Gamma \tag{B.5}$$

The argument of Section 7 is now repeated to show that the right-hand side of (B.5) is non-negative and thereby the minimum principle is established. At the solution we find

$$2J_{\min} = \int_{S_U} \{\dot{\bar{T}}_i \dot{u}_i^0 + \dot{R}_i(D\dot{u}_i^0)\}\, dS - \int_{S_T} \{\dot{\bar{T}}_i^0 \dot{u}_i + \dot{R}_i^0(D\dot{u}_i)\}\, dS \quad \square \tag{B.6}$$

B.2.1. Dual formulation

An alternative dual formulation can be used to obtain $(\dot{\Sigma}, \dot{S})$ uniquely. Introduce the functional

$$F(\dot{r}_I^*, \dot{s}_\alpha^*) \equiv \int_V \left\{\frac{1}{2H}\left(\frac{D_{IJ} r_J^D \dot{r}_I^*}{\Sigma}\right)^2\right\} dV + \int_\Gamma \left\{\frac{1}{2G}\left(\frac{b_{\alpha\beta} s_\beta^D \dot{s}_\alpha^*}{S}\right)^2\right\} d\Gamma - \int_{S_U} \{\dot{\bar{T}}_i^* \dot{u}_i^0 + \dot{R}_i^*(D\dot{u}_i^0)\}\, dS \tag{B.7}$$

Then, $F(\dot{r}_I^*, \dot{s}_\alpha^*)$ is minimized over all equilibrium fields in order to deliver the actual solution. The proof begins with the principle of virtual work (8.4) written in the form

$$\int_V \{\Delta\dot{Q}_{ij}\dot{\varepsilon}_{ij}^{PL} + \Delta\dot{\tau}_{ijk}\dot{\varepsilon}_{ij,k}^{PL}\}\, dV + \int_\Gamma \{\Delta\dot{p}_{ij}[\dot{\varepsilon}_{ij}^{PL}] + \Delta\dot{q}_{ij}\langle\dot{\varepsilon}_{ij}^{PL}\rangle\}\, d\Gamma = \int_S \{\Delta\dot{\bar{T}}_i\dot{u}_i + \Delta\dot{R}_i(D\dot{u}_i)\}\, dS \tag{B.8}$$

where the symbol Δ again denotes the difference between the trial equilibrium field solution (asterisked) and the actual solution (no asterisk). For $\dot{E}^P > 0$ we again find that (7.9) is satisfied, while for $\dot{D} > 0$ we find that (7.10) is satisfied. Consequently we obtain $F(\dot{r}_I, \dot{s}_\alpha) \leqslant F(\dot{r}_I^*, \dot{s}_\alpha^*)$. This completes the proof.

Finally, we note that $F(\dot{r}_I, \dot{s}_\alpha)$ can be written in the form

$$-2F(\dot{r}_I, \dot{s}_\alpha) = \int_{S_U} \{\dot{\bar{T}}_i\dot{u}_i^0 + \dot{R}_i(D\dot{u}_i^0)\}\, dS - \int_{S_T} \{\dot{\bar{T}}_i^0\dot{u}_i + \dot{R}_i^0(D\dot{u}_i)\}\, dS \tag{B.9}$$

and so we conclude that the values of $2J$ in the extremum of (B.3) and of $-2F$ in the extremum of (B.9) approximate the quantity $\int_{S_U}\{\dot{\bar{T}}_i\dot{u}_i^0 + \dot{R}_i(D\dot{u}_i^0)\}\, dS - \int_{S_T}\{\dot{\bar{T}}_i^0\dot{u}_i + \dot{R}_i^0(D\dot{u}_i)\}$ from above and below.

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