Strain gradient plasticity under non-proportional loading

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Abstract

A critical examination is made of two classes of strain gradient plasticity theories currently available for studying micron scale plasticity. One class is characterized by certain stress quantities expressed in terms of increments of strains and their gradients, while the other class employs incremental relations between all stress quantities and the increments of strains and their gradients. Implications stemming from the differences in formulation of the two classes of theories are explored for two basic examples having non-proportional loading: (i) a layer deformed into the plastic range by tensile stretch with no constraint on plastic flow at the surfaces followed by further stretch with plastic flow constrained at the surfaces; and (ii) a layer deformed into the plastic range by tensile stretch followed by bending. The marked difference in predictions by the two theories suggests that critical experiments will be able to distinguish between them.

Keywords: Strain gradient plasticity, non-proportional loading, rate-independent plasticity, rate-dependent plasticity

1. Introduction

The first strain gradient theories of plasticity were proposed over two decades ago (Aifantis, 1984; Fleck and Hutchinson, 1994). An early objective was to extend the classical isotropic hardening theory of plasticity, J_2 flow theory, by incorporating a dependence on gradients of plastic strain. This has turned out to be more difficult than was first anticipated. An otherwise attractive formulation by Fleck and Hutchinson (2001) was found, under some non-proportional straining histories, to violate the thermodynamic requirement that plastic dissipation must be positive. Gudmundson (2004) and Gurtin and Anand (2009), who noted this violation, proposed an alternate formulation which ensured that the thermodynamic dissipation requirement was always met. The manner in which these authors circumvented the problem was unusual for a rate-independent solid—they proposed a constitutive relation in which certain stress quantities are expressed in terms of increments of strain. This formulation admits the possibility of finite stress changes due to infinitesimal changes in strain under non-proportional straining. By contrast, the constitutive relation proposed by Fleck and Hutchinson (2001) was incremental in nature with increments of all stress quantities expressed in terms of increments of strain. This constitutive relation has been modified so that it now satisfies the thermodynamic requirements (Hutchinson, 2012). The consequences of the two classes of formulations for problems involving distinctly non-proportional loading histories will be investigated in this paper. Here, in the interest of brevity and for lack of a better terminology, a constitutive construction in the class proposed by Gudmundson and Gurtin & Anand will be referred to as non-incremental, while that proposed by Fleck and Hutchinson will be termed incremental.

To bring out the differences in predictions for the two classes of theories it is essential to consider problems with non-proportional loading, yet to our knowledge no such studies have been made. Non-proportional loading has played a central role in the history of plasticity not only because it arises in applications but by serving to clarify critical aspects of constitutive behavior. Under nearly proportional histories, the predictions of the two theories differ only slightly. Indeed, the two formulations employed in this paper coincide with the prediction of a deformation theory of strain gradient plasticity under strictly proportional straining histories. The deformation theory is a nonlinear elasticity theory devised to mimic elastic-plastic behavior under monotonic loading for problems with little or no departure from proportional straining. Almost all investigations in the literature employing strain gradient plasticity, whether based on the incremental or the non-incremental formulation, have focused on problems with loads applied proportionally. Here, two basic non-proportional loading problems are studied. The first is a layer of material stretched uniformly in plane strain tension into the plastic range with no constraint on plastic flow at its surfaces. Then, at a prescribed stretch, plastic flow is constrained such that no further plastic strain occurs at the surfaces as the layer undergoes further stretch. The constraint models passivation of the surfaces at the prescribed stretch whereby a very thin layer is deposited on the surface blocking dislocations from passing out of the surface. The second problem is again a layer stretched uniformly in plane strain tension into the plastic range to a prescribed stretch at which point bending is imposed on the layer with no additional average stretch. Non-proportionality arises due to the abrupt change in the distribution of the strain-rate caused by passivation or by switching from stretching to bending.

For each example, the most important aspects of the predictions of the two theories are illustrated and contrasted. The calculations involved in these examples expose some interesting and unusual mathematical aspects of the non-incremental theories; these are identified and analyzed. The paper is organized as follows. Section 2 introduces specific versions of the two classes of theories together with the deformation theory with which they coincide for proportional straining. Section 3 deals with the two plane strain problems, stretch-passivation and stretch-bend. Section 4 presents a detailed analysis of mathematical aspects of the non-incremental theory for the stretch-passivation problem with further details given in the Appendix. Finally, in Section 5, an overview summary is presented of both the mathematical and physical findings from this study. Differences in the predictions of the two classes of theories that have significant physical implications are highlighted.

2. The two classes of strain gradient plasticity

The established small strain framework for strain gradient plasticity will be adopted (Muhlhaus & Aifantis 1991; Fleck & Hutchinson 1997; Gurtin & Anand 2005). Equality of the internal and external virtual work is

$$\int_V \left\{ \sigma_{ij}\delta\varepsilon_{ij}^e + q_{ij}\delta\varepsilon_{ij}^P + \tau_{ijk}\delta\varepsilon_{ij,k}^P \right\} dV = \int_S \left(T_i \delta u_i + t_{ij}\delta\varepsilon_{ij}^P \right) dS \qquad (2.1)$$

with volume of the solid V, surface S, displacements u_i, total strains $\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2$, plastic strains ε_{ij}^P ($\varepsilon_{kk}^P = 0$), and elastic strains $\varepsilon_{ij}^e = \varepsilon_{ij} - \varepsilon_{ij}^P$. The symmetric Cauchy stress is σ_{ij}, and the stress quantities work conjugate to increments of ε_{ij}^P and $\varepsilon_{ij,k}^P$ are q_{ij} ($q_{ij} = q_{ji}$ & $q_{kk} = 0$) and τ_{ijk} ($\tau_{ijk} = \tau_{jik}$ & $\tau_{jjk} = 0$). The surface tractions are $T_i = \sigma_{ij} n_j$ and $t_{ij} = \tau_{ijk} n_k$ with n_i as the outward unit normal to S. The equilibrium equations are

$$\sigma_{ij,j} = 0, \quad -s_{ij} + q_{ij} - \tau_{ijk,k} = 0 \qquad (2.2)$$

with $s_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$.

Isotropic elastic behavior will be assumed with elastic moduli L_{ijkl}^e such that $\sigma_{ij} = L_{ijkl}^e \varepsilon_{kl}^e$. A generalized effective plastic strain is defined as

$$E_P = \sqrt{{\varepsilon_P}^2 + \ell^2 \varepsilon_P^{*2}} \quad \text{with} \quad \varepsilon_P = \sqrt{2\varepsilon_{ij}^P \varepsilon_{ij}^P / 3} \quad \text{and} \quad \varepsilon_P^* = \sqrt{2\varepsilon_{ij,k}^P \varepsilon_{ij,k}^P / 3} \tag{2.3}$$

with ℓ as the single material length parameter. This definition is a special case of a family of isotropic measures of the plastic strain gradients defined by Fleck and Hutchinson (1997, 2001), and it adequately serves the purpose of the present study. For the two theories introduced in this paper the only other input characterizing the material is the relation between the stress and the effective plastic strain in uniaxial tension, $\sigma_0(\varepsilon_P)$, which is assumed to be monotonically increasing with $\sigma_Y = \sigma_0(0)$ as the initial tensile yield stress.

We begin by defining the *deformation theory version of strain gradient plasticity*. The deformation theory will be used as the template for the two theories used in this paper by defining them such that they each coincide with the deformation theory for proportional straining. The deformation theory is version of small strain, nonlinear elasticity, with energy density dependent on ε_{ij} and ε_{ij}^P. Specifically, following Fleck and Hutchinson (1997, 2001), take as the energy density

$$\psi(\varepsilon_{ij}, \varepsilon_P) = \frac{1}{2} L_{ijkl}^e \left(\varepsilon_{ij} - \varepsilon_{ij}^P\right)\left(\varepsilon_{kl} - \varepsilon_{kl}^P\right) + U_P(E_P) \quad \text{with} \quad U_P(E_P) = \int_0^{E_P} \sigma_0(\varepsilon_P) d\varepsilon_P \tag{2.4}$$

where the constraint, $\varepsilon_{ij}^P = \varepsilon_P m_{ij}$, is imposed with $m_{ij} = 3s_{ij} / (2\sigma_e)$ and $\sigma_e = \sqrt{3s_{ij}s_{ij} / 2}$. The stresses generated are

$$\begin{gathered} \sigma_{ij} = \frac{\partial \psi}{\partial \varepsilon_{ij}^e} = L_{ijkl}^e \varepsilon_{kl}^e , \\ q_{ij} = \frac{\partial \psi}{\partial \varepsilon_{ij}^P} = \frac{2}{3} \sigma_0(E_P) \frac{\varepsilon_{ij}^P}{E_P}, \quad \tau_{ijk} = \frac{\partial \psi}{\partial \varepsilon_{ij,k}^P} = \frac{2}{3} \ell^2 \sigma_0(E_P) \frac{\varepsilon_{ij,k}^P}{E_P} \end{gathered} \tag{2.5}$$

The stress-strain behavior input, $\sigma_0(\varepsilon_P)$, is reproduced when (2.5) is specialized to uniaxial tension. The potential energy of a body is

$$F(u_i, \varepsilon_P) = \int_V \psi(\varepsilon_{ij}, \varepsilon_{ij}^P) dV - \int_{S_T} \left\{T_i u_i + t\varepsilon_P\right\} dS \tag{2.6}$$

with prescribed T_i and $t = t_{ij} m_{ij}$ on portions of the surface, S_T, and with u_i and ε_P prescribed on the remaining surface S_U. The solution to the boundary value problem minimizes the potential energy among all admissible u_i and ε_P.

The notion of *proportional plastic straining* will be important in the sequel. Within the context of strain gradient plasticity, proportional straining histories are the limited set for which the plastic strains and their gradients increase in proportion according to

$$\left(\varepsilon_{ij}^{P}, \varepsilon_{ij,k}^{P}\right) = \lambda\left((\varepsilon_{ij}^{P})^{0}, (\varepsilon_{ij,k}^{P})^{0}\right) \tag{2.7}$$

with λ increasing monotonically, and quantities with superscript "0" independent of λ.

2.1 Non-incremental theories with certain stresses expressed in terms of strain increments

The Cauchy stress continues to be given by $\sigma_{ij} = L_{ijkl}^{e} \varepsilon_{kl}^{e}$. The prescription for defining the higher order stresses, q_{ij} and τ_{ijk}, follows the idea proposed by Gudmundson (2004) and Gurtin and Anand (2009) who were motivated to ensure that the dissipative plastic work rate is never negative. For the purposes of this paper, it will suffice to construct a version of this class of theories with unrecoverable plastic work—the notation q_{ij}^{UR} and τ_{ijk}^{UR} will be employed to indicate this. In the terminology of Gurtin and Anand (2005, 2009), these higher order stresses are entirely dissipative. To this end, define generalized stress and plastic strain-rate vectors according to

$$\mathbf{\Sigma} = \sqrt{3/2}\left(q_{ij}^{UR}, \ell^{-1}\tau_{ijk}^{UR}\right), \quad \dot{\mathbf{E}}_P = \sqrt{2/3}\left(\dot{\varepsilon}_{ij}^{P}, \ell\dot{\varepsilon}_{ij,k}^{P}\right) \tag{2.8}$$

such that the plastic work rate is

$$q_{ij}^{UR}\dot{\varepsilon}_{ij}^{P} + \tau_{ijk}^{UR}\dot{\varepsilon}_{ij,k}^{P} = \mathbf{\Sigma}\bullet\dot{\mathbf{E}}_P \tag{2.9}$$

The vector magnitudes are

$$\Sigma = \left|\mathbf{\Sigma}\right| = \sqrt{\tfrac{3}{2}q_{ij}^{UR}q_{ij}^{UR} + \tfrac{3}{2}\ell^{-2}\tau_{ijk}^{UR}\tau_{ijk}^{UR}}\ , \quad \dot{\mathrm{E}}_P = \left|\dot{\mathbf{E}}_P\right| = \sqrt{\tfrac{2}{3}\dot{\varepsilon}_{ij}^{P}\dot{\varepsilon}_{ij}^{P} + \tfrac{2}{3}\ell^{2}\dot{\varepsilon}_{ij,k}^{P}\dot{\varepsilon}_{ij,k}^{P}} \tag{2.10}$$

Let $\mathrm{E}_P = \int \dot{\mathrm{E}}_P dt$ where t is time, and note that this monotonically increasing measure of the effective plastic strain is defined differently from E_P in (2.3). The latter is not monotonic and is zero when the plastic strain and its gradient vanish. The two measures coincide for proportional plastic straining. In the absence of plastic strain gradients, or if $\ell = 0$, E_P reduces to the effective plastic strain used in conventional J_2 flow theory, $e_P = \int \sqrt{\frac{2}{3}\dot{\varepsilon}_{ij}^P \dot{\varepsilon}_{ij}^P}$. In the present paper the distinction between e_P, which is non-decreasing, and ε_P defined in (2.3), which can increase or decrease, is important and analogous to the distinction between E_P and E_P.

The construction of Gudmundson (2004) and Gurtin and Anand (2005, 2009) specifies $\mathbf{\Sigma}$ to be co-directional to $\dot{\mathbf{E}}_P$ such that, by (2.9), the plastic dissipation rate is never negative. Here, the specific choice adopted by Fleck and Willis (2009a,b) in their study of this class of theories will be used:

$$\mathbf{\Sigma} = \sigma_0(\mathrm{E}_P)\frac{\dot{\mathbf{E}}_P}{\dot{\mathrm{E}}_P}\text{, or,}$$

$$q_{ij}^{UR} = \frac{2}{3}\sigma_0(\mathrm{E}_P)\frac{\dot{\varepsilon}_{ij}^P}{\dot{\mathrm{E}}_P}, \quad \tau_{ijk}^{UR} = \frac{2}{3}\ell^2\sigma_0(\mathrm{E}_P)\frac{\dot{\varepsilon}_{ij,k}^P}{\dot{\mathrm{E}}_P} \tag{2.10}$$

This choice coincides with the deformation theory (2.5) for proportional straining and reduces to J_2 flow theory when $\ell = 0$. A change in the direction of loading can lead to a finite change in the distribution $\dot{\varepsilon}_{ij}^P$ and its gradient. When this occurs, by (2.10), $\mathbf{\Sigma}$ can undergo finite changes. In other words, an infinitesimal change in loads on the boundary of the solid can produce finite changes in q_{ij}^{UR} and τ_{ijk}^{UR}. The stretch-passivation problem analyzed later provides an example of such behavior. It is largely the potential consequences of the constitutive assumption embodied in (2.10) which motivates this study. The findings also have implications for strain gradient theories of single crystals which based on the same constitutive construction.

Using the definitions in (2.8) and (2.10), one finds $\Sigma = \sigma_0(\mathrm{E}_P)$. In the class of theories introduced above normality exists in the sense that $\dot{\mathbf{E}}_P$ is normal to the surface in the generalized stress space specified by $\Sigma = \sigma_0(\mathrm{E}_P)$ (Fleck & Willis, 2009a,b). However, the correct interpretation is that $\mathbf{\Sigma}$ locates itself on this surface depending on $\dot{\mathbf{E}}_P$, because $\mathbf{\Sigma}$ is defined in terms of $\dot{\mathbf{E}}_P$ and not vice versa. As Fleck and Willis have emphasized, the components of $\mathbf{\Sigma}$ are not fixed in the current state. They depend on the current strain-rates which in turn depend on the prescribed incremental boundary conditions. This is analogous to conventional stresses in the theory of a rigid-plastic solid for which $\mathbf{\Sigma}$ remains on the yield surface but its components undergo finite changes when directional changes in $\dot{\mathbf{E}}_P$ occur.

Fleck and Willis (2009a,b) derived two coupled minimum principles governing the incremental boundary value problem for this class of theories. In the current state, E_P and σ_{ij} are known but $\mathbf{\Sigma}$ is not known. Minimum principle I is used to determine the spatial distribution of $\dot{\varepsilon}_{ij}^P$ (and $\mathbf{\Sigma}$) while principle II determines $\dot{u}_i$ and the amplitude of the plastic strain rate field if it has not been determined by principle I. The following statements suffice for the examples considered in this paper for which the tractions, t_{ij}, when prescribed on a surface, are taken to be zero and $\dot{\varepsilon}_{ij}^P$, when prescribed on a surface, are also taken to be zero. For other sets of boundary conditions and for full details, the reader is referred to the Fleck-Willis papers. Consider all admissible distributions $\dot{\varepsilon}_{ij}^P$ satisfying $\dot{\varepsilon}_{ij}^P = 0$ on portions of the surface where it is prescribed. Apart from a possible amplitude factor, the actual distribution minimizes

$$\Phi_I = \int_V \left(\sigma_0(\mathrm{E}_P)\dot{\mathrm{E}}_P - s_{ij}\dot{\varepsilon}_{ij}^P \right) dV \quad \text{with} \quad \left(\Phi_I\right)_{MIN} = 0 \tag{2.11}$$

where, on the portions of the surface on which $\dot{\varepsilon}_{ij}^P$ is unconstrained, $t_{ij} = 0$. Under these conditions, the amplitude of the distribution is undetermined and a normalizing constraint on the distribution of $\dot{\varepsilon}_{ij}^P$ must be added. Minimum principle II states that

$$\Phi_{II} = \frac{1}{2}\int_V \left(L_{ijkl}^e (\dot{\varepsilon}_{ij} - \dot{\varepsilon}_{ij}^P)(\dot{\varepsilon}_{ij} - \dot{\varepsilon}_{ij}^P) + \frac{d\sigma_0(\mathrm{E}_P)}{d\mathrm{E}_P}\dot{\mathrm{E}}_P{}^2 \right) dV - \int_{S_T} \dot{T}_i \dot{u}_i dS \tag{2.12}$$

is minimized by the solution $\dot{u}_i$ and the amplitude of the plastic strain rate field.

Rate-dependent versions of this class of theories have proved to be relatively straightforward to implement in numerical codes and widely adopted. To illustrate the influence of the rate-dependence on the issue of non-proportional loading, we will present some results based on the following standard incorporation of time dependence following Fleck and Willis (2009a,b). Let $\dot{\varepsilon}_R$ be a reference strain-rate. Introduce the following potential of the plastic strain-rates:

$$\varphi(\dot{\mathrm{E}}_P) = \frac{\sigma_0(\mathrm{E}_P)\dot{\varepsilon}_R}{1+m}\left(\frac{\dot{\mathrm{E}}_P}{\dot{\varepsilon}_R}\right)^{1+m} \tag{2.13}$$

where m is the strain-rate exponent which delivers the rate-independent limit when $m \to 0$. The associated stress quantities are

$$q_{ij} = \frac{\partial\varphi}{\partial\dot{\varepsilon}_{ij}^P} = \frac{2}{3}\sigma_0(\mathrm{E}_P)\left(\frac{\dot{\mathrm{E}}_P}{\dot{\varepsilon}_R}\right)^m \frac{\dot{\varepsilon}_{ij}^P}{\dot{\mathrm{E}}_P}, \quad \tau_{ijk} = \frac{\partial\varphi}{\partial\dot{\varepsilon}_{ij,k}^P} = \frac{2}{3}\ell^2\sigma_0(\mathrm{E}_P)\left(\frac{\dot{\mathrm{E}}_P}{\dot{\varepsilon}_R}\right)^m \frac{\dot{\varepsilon}_{ij,k}^P}{\dot{\mathrm{E}}_P} \tag{2.14}$$

These expressions are identical to the rate-independent expressions in (2.10) apart from the factor $\left(\dot{\mathrm{E}}_P / \dot{\varepsilon}_R\right)^m$. For rate-dependent problems, the only change to minimum principle I in (2.11) is that $\sigma_0(\mathrm{E}_P)\dot{\mathrm{E}}_P$ is replaced by $\varphi(\dot{\mathrm{E}}_P)$. For rate-dependent problems, the plastic strain rate is fully determined by minimizing Φ_I. The rate-dependent form of principle II will not be needed in the examples considered in this paper.

4.2 Incremental theories with stress increments expressed in terms of strain increments

In this class of theories, normality of the plastic strain rate to the conventional J_2 yield surface is retained with $\dot{\varepsilon}_{ij}^P = \dot{e}_P m_{ij}$ where $m_{ij} = 3s_{ij} / 2\sigma_e$ and $\sigma_e = \sqrt{3s_{ij}s_{ij} / 2}$ such that

$$\dot{e}_P = \sqrt{2\dot{\varepsilon}_{ij}^P\dot{\varepsilon}_{ij}^P / 3} \ge 0 \tag{2.15}$$

The specification adopted is a modification of Fleck-Hutchinson (2001) theory outlined in Hutchinson (2012) such that the dissipative contribution is always non-negative. Recoverable contributions are derived from the free energy function

$$\psi(\varepsilon_{ij},\varepsilon_{ij}^P)=\frac{1}{2}L_{ijkl}^e\left(\varepsilon_{ij}-\varepsilon_{ij}^P\right)\left(\varepsilon_{kl}-\varepsilon_{kl}^P\right)+U_P(E_P)-U_P(\varepsilon_P) \tag{2.16}$$

with E_P and ε_P defined in (2.3) and U_P in (2.4). The contribution of the plastic strains and their gradients to the free energy, $\psi_P=U_P(E_P)-U_P(\varepsilon_P)$, vanishes when the gradients vanish and is otherwise non-negative. The recoverable stresses generated from (2.16) are

$$\begin{aligned}
&\sigma_{ij}=\frac{\partial\psi}{\partial\varepsilon_{ij}^e}=L_{ijkl}^e\varepsilon_{kl}^e\,,\\
&q_{ij}^R=\frac{\partial\psi}{\partial\varepsilon_{ij}^P}=\frac{2}{3}\sigma_0(E_P)\frac{\varepsilon_{ij}^P}{E_P}-\frac{2}{3}\sigma_0(\varepsilon_P)\frac{\varepsilon_{ij}^P}{\varepsilon_P},\quad \tau_{ijk}^R=\frac{\partial\psi}{\partial\varepsilon_{ij,k}^P}=\frac{2}{3}\ell^2\sigma_0(E_P)\frac{\varepsilon_{ij,k}^P}{E_P}
\end{aligned} \tag{2.17}$$

The unrecoverable plastic work is taken to be the same as in conventional J_2 flow theory:

$$U_P(e_P)=\int_0^{e_P}\sigma_0(e_P)de_P \quad \text{with} \quad e_P=\int\dot{e}_p \tag{2.18}$$

By (2.15), $U_P(e_P)$ is non-decreasing. The unrecoverable stress components are $q_{ij}^{UR}=\sigma_0(e_P)m_{ij}$ with $\tau_{ijk}^{UR}=0$ such that the dissipative plastic work rate is always non-negative: $q_{ij}^{UR}\dot{\varepsilon}_{ij}^P=\dot{U}_P\geq 0$. The complete set of stresses is: σ_{ij}, $q_{ij}=q_{ij}^R+q_{ij}^{UR}$ and $\tau_{ijk}=\tau_{ijk}^R$. Under proportional straining with m_{ij} fixed, $\sigma_0(e_P)m_{ij}=2\sigma_0(\varepsilon_P)\varepsilon_{ij}^P/3\varepsilon_P$, and all the stresses coincide with those in (2.5) for deformation theory. In addition, the theory reduces to conventional J_2 flow theory in the limit $\ell\to 0$. Thus, both classes of theories introduced and used in this paper coincide with the deformation theory for proportional straining and both reduce to J_2 flow theory when $\ell\to 0$. If gradient effects are important, significant differences between the two theories arise under distinctly non-proportional straining, as illustrated in this paper.

The minimum principle for the incremental boundary value problem for this theory is similar in structure to that for conventional J_2 flow theory except that it brings in gradients of the plastic strain rate. The principle requires the quadratic functional F to be minimized with respect $\dot{u}_i$ and $\dot{e}_p$ where

$$F(\dot{u}_i, \dot{e}_P) = \int_V \varphi(\dot{\varepsilon}_{ij}, \dot{e}_P) dV - \int_{S_T} \left(\dot{T}_i \dot{u}_i + \dot{t} \dot{e}_P \right) dS \quad \text{with} \quad \varphi = \frac{1}{2} \left(\dot{\sigma}_{ij} \dot{\varepsilon}_{ij}^e + \dot{q}_{ij} \dot{\varepsilon}_{ij}^P + \dot{\tau}_{ijk} \dot{\varepsilon}_{ij,k}^P \right) \tag{2.19}$$

with $\dot{e}_P \geq 0$, $(\dot{T}_i,\ \dot{t})$ prescribed on S_T, and $(\dot{u}_i, \dot{e}_p)$ prescribed on S_U. A direct calculation gives

$$\begin{aligned} 2\varphi(\dot{\varepsilon}_{ij}, \dot{e}_P) &= L_{ijkl}^e (\dot{\varepsilon}_{ij} - \dot{\varepsilon}_{ij}^P)(\dot{\varepsilon}_{ij} - \dot{\varepsilon}_{ij}^P) \\ &+ S(E_P)\dot{E}_P{}^2 - S(\varepsilon_P)\dot{\varepsilon}_P{}^2 + \frac{\sigma_0(E_P)}{E_P} \ddot{\mathrm{E}}_P{}^2 - \frac{\sigma_0(\varepsilon_P)}{\varepsilon_P} \dot{e}_P{}^2 + \frac{d\sigma_0(e_P)}{de_P} \dot{e}_P{}^2 \end{aligned} \tag{2.20}$$

where $S(\varepsilon) \equiv d\sigma_0(\varepsilon) / d\varepsilon - \sigma_0(\varepsilon) / \varepsilon$. Because $\dot{\varepsilon}_{ij}^P = \dot{e}_P m_{ij}$, it follows that $\dot{\varepsilon}_P = 2\dot{e}_P m_{ij} \varepsilon_{ij}^P / 3\varepsilon_P$, $\ddot{\mathrm{E}}_P{}^2 = \dot{e}_P{}^2 \left(1 + 2\ell^2 m_{ij,k} m_{ij,k} / 3\right) + \ell^2 \dot{e}_{P,k} \dot{e}_{P,k}$, and $\dot{E}_P = 2\left\{ \dot{e}_P \left(m_{ij}\varepsilon_{ij}^P + \ell^2 m_{ij,k} \varepsilon_{ij,k}^P \right) + \dot{e}_{P,k} \ell^2 m_{ij} \varepsilon_{ij,k}^P \right\} / 3E_P$. These permit (2.20) to be re-assembled as a convex function of $(\dot{\varepsilon}_{ij}, \dot{e}_P, \dot{e}_{P,i})$:

$$2\varphi(\dot{\varepsilon}_{ij}, \dot{e}_P) = L_{ijkl}^e (\dot{\varepsilon}_{ij} - \dot{\varepsilon}_{ij}^P)(\dot{\varepsilon}_{ij} - \dot{\varepsilon}_{ij}^P) + C\dot{e}_P{}^2 + C_i \dot{e}_P \dot{e}_{P,i} + C_{ij} \dot{e}_{P,i} \dot{e}_{P,j} \tag{2.21}$$

where the C's depend on the current distribution of plastic strain, m_{ij}, and ℓ^2.

The yield condition for this theory (Hutchinson, 2012) is based on the Cauchy stress: $\sigma_e \equiv \sqrt{3 s_{ij} s_{ij} / 3} = \sigma_Y$. Prior to plastic straining, $\sigma_Y = \sigma_0(0)$. During plastic straining, $\dot{e}_P > 0$, σ_Y is updated by $\dot{\sigma}_Y = \dot{\sigma}_e$. This is similar to the conventional J_2 flow theory yielding condition, but it differs in that $\dot{\sigma}_Y$ can be positive or negative depending on the strain rate gradient. The choice of yield condition is consistent with the normality condition previously introduced, i.e., $\dot{\varepsilon}_{ij}^P = \dot{e}_P m_{ij}$. A rate-dependent version of this theory can also be introduced, but it is not needed in this paper.

The examples in this paper take the elastic response to be isotropic and incompressible with Young's modulus E. For both theories, the input tensile curve is $\sigma_0(\varepsilon_P) = \sigma_Y \left(1 + k\varepsilon_P{}^N\right)$, with initial yield stress, $\sigma_Y = \sigma_0(0)$, and yield strain $\varepsilon_Y = \sigma_Y / E$. In dimensionless form,

$$\frac{\sigma_0(\varepsilon_P)}{\sigma_Y} = \left(1 + p\left(\frac{\varepsilon_P}{\varepsilon_Y} \right)^N \right) \quad \text{with} \quad p = k\varepsilon_Y{}^N \tag{2.22}$$

3. Two plane strain problems for an infinite layer

Non-proportional conditions in this section are created for an initially uniform layer of thickness $2h$ undergoing plane strain tension by abruptly changing the constraint on plastic flow at the top and bottom surfaces of the layer or by abruptly switching from stretching to bending. By constraining the plastic strain rate to vanish at the surfaces, one can model the effect of surface passivation which blocks dislocation motion across the surfaces. In the first example, it is imagined that surface passivation is done under load following unconstrained plastic straining. Passivation blocks additional plastic flow at the surfaces. This relatively simple example provides insights into basic aspects of the behavior predicted by the two classes of models. Even though plane strain conditions prevail throughout, non-proportionality arises due to the abrupt change in plastic strain-rate distribution across the layer, altering the ratio of the gradient of plastic strain rate to the plastic strain rate itself. In the second example, the layer is stretched uniformly into the plastic range and then, with no further overall stretch, is subject to pure bending. The surfaces are unconstrained in throughout the entire history such that gradients of plastic flow and non-proportionality arise owing to the switch from stretch to bending.

The layer occupies $-h \le x_1 \le h$ and is stretched along the x_1-direction, and is subject to $u_3 = 0$. Under these conditions, the total strains are uniform if there is no bending or vary linearly if bending occurs, with only two non-zero components: $\varepsilon_{22} = -\varepsilon_{11}$. The non-zero plastic strain components are $\varepsilon_{22}^P(x_2) = -\varepsilon_{11}^P(x_2)$, with $\varepsilon_P = 2\left|\varepsilon_{11}^P\right| / \sqrt{3}$, $\varepsilon_P^* = \left|d\varepsilon_P / dx_2\right|$ and $\dot{e}_P = 2\left|\dot{\varepsilon}_{11}^P\right| / \sqrt{3}$. The non-zero stress quantities are $\sigma_{33} = \sigma_{11} / 2$, $s_{22} = -s_{11} = -\sigma_{11} / 2$, $\sigma_e = \sqrt{3}\left|\sigma_{11}\right| / 2$, $q_{22} = -q_{11}$ and $\tau_{222} = -\tau_{112}$. These stress components are functions only of x_2 and the equilibrium equations in (2.2) are satisfied except for $-s_{11} + q_{11} - \tau_{112,2} = 0$.

The boundary conditions on the top and bottom surfaces will have $T_i = 0$ in all cases and either *constrained plastic flow*, $\dot{\varepsilon}_{11}^P = 0$ (with $\dot{\tau}_{112} \ne 0$ and $\dot{t}_{11} \ne 0$), or *unconstrained plastic flow*, $\dot{t}_{11} = 0$ and $\dot{\tau}_{112} = 0$ (with $\dot{\varepsilon}_{11}^P \ne 0$). Thus, for all the problems considered in this section, there is no traction work done on the layer at its surfaces. The load will be applied by imposing overall stretch, ε_{11}^0, and/or bending curvature, κ, such that the strain in the layer is $\varepsilon_{11} = \varepsilon_{11}^0 + \kappa x_2$. Results will be presented for the average tensile stress in the layer, and for the bending moment/depth in the second problem. For the surface conditions assumed, these are given for both theories by

$$\bar{\sigma}_{11} = \frac{1}{2h}\int_{-h}^{h} \sigma_{11} dx_2 = \frac{2E}{3h}\int_{-h}^{h}\left(\varepsilon_{11} - \varepsilon_{11}^{P}\right)dx_2 \,, \quad M = \frac{4E}{3}\int_{-h}^{h}\left(\varepsilon_{11} - \varepsilon_{11}^{P}\right)x_2 dx_2 \tag{3.1}$$

Minimum principles for the theories introduced in Section 2 follow directly. Because ε_{11} will be prescribed, only the distribution of $\varepsilon_{11}^{P}(x_2)$ is unknown subject to either full constraint, $\varepsilon_{11}^{P} = 0$, or no constraint at the surfaces. For the non-incremental theory, the Fleck-Willis minimum principles, (2.11) and (2.12), reduce to (for a unit length of layer)

$$\Phi_I = \int_{-h}^{h}\left(\sigma_0(\mathrm{E}_P)\dot{\mathrm{E}}_P - \sigma_{11}\dot{\varepsilon}_{11}^{P}\right)dx_2 \quad \text{with} \quad \left(\Phi_I\right)_{MIN} = 0 \tag{3.2}$$

and

$$\Phi_{II} = \frac{1}{2}\int_{-h}^{h}\left(\frac{4E}{3}\left(\dot{\varepsilon}_{11} - \dot{\varepsilon}_{11}^{P}\right)^2 + \frac{d\sigma_0(\mathrm{E}_P)}{d\mathrm{E}_P}\dot{\mathrm{E}}_P{}^2\right)dx_2 \tag{3.3}$$

where $\sigma_0(\mathrm{E}_P) = \sigma_Y\left(1 + k\mathrm{E}_P{}^N\right)$, $\dot{\mathrm{E}}_P = \sqrt{\dot{e}_P{}^2 + (\ell d\dot{e}_P / dx_2)^2}$ with $\dot{e}_P = 2\left|\dot{\varepsilon}_{11}^{P}\right| / \sqrt{3}$. Conversion for the rate-dependent version is immediate following the prescription discussed in connection with (2.13), i.e., replacing $\sigma_0(\mathrm{E}_P)\dot{\mathrm{E}}_P$ by $\varphi(\dot{\mathrm{E}}_P)$ in (3.2).

The minimum principle (2.19) for the incremental theory becomes, for a unit length of layer,

$$F(\dot{\varepsilon}_{11}, \dot{\varepsilon}_{11}^{P}) = \frac{1}{2}\int_{-h}^{h}\left(\frac{4E}{3}\left(\dot{\varepsilon}_{11} - \dot{\varepsilon}_{11}^{P}\right)^2 + C\dot{e}_P{}^2 + C_2\dot{e}_P\frac{d\dot{e}_P}{dx_2} + C_{22}\left(\frac{d\dot{e}_P}{dx_2}\right)^2\right)dx_2 \tag{3.4}$$

The C s are obtained using expression (2.20). Note that $m_{11} = -m_{22} = \sqrt{3}\,\mathrm{sign}(\sigma_{11})/2$ such that $dm_{11}/dx_2 = 0$; plastic loading requires $m_{11}\dot{\varepsilon}_{11}^{P} > 0$.

3.1 The stretch-passivation problem

The first example considers stretch of the layer into the plastic range with no constraint on plastic flow at the surfaces until $\varepsilon_{11} = \varepsilon_T$ when constraint at the surfaces is switched on (for example, by passivating the surfaces under load) for the subsequent increments of stretch. The boundary conditions in this problem are ones which a strain gradient plasticity theory must be able to handle. The problem has the additional advantage that its mathematical formulation is relatively simple. With no constraint at the surfaces, $\dot{\varepsilon}_{11}^P$ is unconstrained in the minimum principles (3.2) and (3.4), while $\dot{\varepsilon}_{11}^P = 0$ at $x_2 = \pm h$ if constraint is active. Uniform plane strain tension holds for both theories for $\varepsilon_{11} \le \varepsilon_T$. With $\varepsilon_Y = \sigma_Y / E$ plastic yield occurs at $\sigma_{11} = 2\sigma_Y / \sqrt{3}$ or $\varepsilon_{11} = \sqrt{3}\varepsilon_Y / 2$, such that for $\sqrt{3}\varepsilon_Y / 2 \le \varepsilon_{11} \le \varepsilon_T$

$$\frac{\varepsilon_{11}^P}{\varepsilon_Y} = \frac{\sqrt{3}}{2k\varepsilon_Y{}^N}\left(\frac{\sqrt{3}\sigma_{11}}{2\sigma_Y} - 1\right)^{1/N}, \quad \frac{\sigma_{11}}{\sigma_Y} = \frac{4}{3}\left(\frac{\varepsilon_{11}}{\varepsilon_Y} - \frac{\varepsilon_{11}^P}{\varepsilon_Y}\right), \quad (q_{11} = s_{11} = \sigma_{11}/2, \quad \tau_{112} = 0) \qquad (3.5)$$

Consider first the non-incremental theory. For the first increment after passivation at $\varepsilon_{11} = \varepsilon_T$, $\Phi_I = \sqrt{3}(\sigma_{11}/2)\int_{-h}^{h}\left(\dot{\mathrm{E}}_P - \dot{e}_P\right)dx_2$ by (3.2). It is easily seen that the minimum of Φ_I, among all $\dot{e}_P \ge 0$ subject to $\dot{e}_P = 0$ at $x_2 = \pm h$, is $\dot{e}_P = 0$. Thus, this theory predicts that no plasticity occurs in the first increment of stretch following the imposition of surface constraint. The layer undergoes a uniform incremental elastic response for a finite interval of stretch beyond ε_T. As long as no additional plastic strain occurs, principle I minimizes

$$\Phi_I = (\sqrt{3}/2)\int_{-h}^{h}\left(\sigma_{11}^T\dot{\mathrm{E}}_P - \sigma_{11}\dot{e}_P\right)dx_2 \quad \text{with} \quad \dot{e}_P \ge 0 \quad \text{and} \quad \dot{e}_P(\pm h) = 0 \qquad (3.6)$$

where σ_{11} is uniform and σ_{11}^T denotes the value of σ_{11} at $\varepsilon_{11} = \varepsilon_T$. Plastic straining resumes when the stress σ_{11} becomes large enough such that a non-zero solution $\dot{e}_P$ exists minimizing (3.6) with $\Phi_I = 0$. This is an eigenvalue problem for $\sigma_{11} \equiv \sigma_{11}^C$. Let $R = \sigma_{11} / \sigma_{11}^T$, be the normalized eigenvalue, divide (3.6) by $\sqrt{3}\sigma_{11}^T / 2$, and let $y(x) = \dot{e}_P(x_2)$ with $x = x_2 / h$ and $(\ )' = d(\ )/dx_2$ to obtain

$$\bar{\Phi}_I(y)=\int_{-h}^{h}\left(\sqrt{\left((\ell y')^2+y^2\right.}-Ry\right)dx_2 \text{ with } y(x_2)\ge 0 \text{ and } y(\pm h)=0 \tag{3.7}$$

There are interesting and fundamental mathematical issues associated with this eigenvalue problem. Section 4 is devoted to the analysis of the eigenvalue problem along with other mathematical issues related to the early stages after the resumption of plastic flow. There is only one possible candidate eigenvalue, $R=R_C>1$, plotted in Fig. 1a. The associated solution $y(x_2)$ (with $y(0)=1$) is plotted in Fig. 1. It has the undesirable property that $y(\pm h)\neq 0$. Thus, strictly, the only acceptable solution is $y(x_2)=0$. Computations with admission of small rate-dependence (Fig. 2) nevertheless strongly suggest that plastic flow resumes at $R=R_C$. The eigenvalue problem will be discussed fully in Section 4.

The implication of the results in Fig. 1a is that the class of theories with non-incremental stresses predicts a significant delay in the resumption of plastic flow following passivation. This delay is also evident in the predictions from the rate-dependent version of the theory, as seen in the example in Fig. 2. For the lowest strain-rate sensitivity ($m=0.01$) and $\ell/h=0.2$, approximately a 10% increase of stress above the stress at passivation is predicted to occur with essentially no plastic straining. This elastic gap is similar to that predicted by the eigenvalue problem for the rate-independent limit for $\ell/h=0.2$.

The *incremental theory* predicts no elastic gap in plastic straining following passivation, only reduced plastic straining. Specifically, for the first increment following passivation, the solution to minimum principle (3.4) can be obtained analytically with the result

$$\frac{\dot{\varepsilon}_{11}^P(x_2)}{\dot{\varepsilon}_{11}}=K\left(1-\frac{\cosh(\beta x_2/\ell)}{\cosh(\beta h/\ell)}\right) \text{ with } K=\frac{E}{E+\left(\dfrac{d\sigma_0}{d\varepsilon_P}\right)_{\varepsilon_P^T}} \quad \& \ \beta=\sqrt{\frac{E}{K}\left(\frac{\varepsilon_P}{\sigma_0(\varepsilon_P)}\right)_{\varepsilon_P^T}} \tag{3.8}$$

Had no passivation occurred, $\dot{\varepsilon}_{11}^P=K\dot{\varepsilon}_{11}$, and thus the reduction in the plastic strain increment and the non-uniformity due to passivation is reflected by the hyperbolic cosine dependence in (3.8). The plot in Fig. 3 shows the full response following passivation for the same problem considered for the non-incremental theory, but generated by solving sequentially, increment by increment, the minimum principle (3.4) for the rate-independent problem. The distinctly different behavior following passivation is evident in Fig. 4 where results for the two theories are directly compared. This difference will be revisited at the end of paper.

3.2 Stretch-bend with no constraint of plastic flow at the surfaces

The problem considered has no constraint on plastic flow at the surfaces at any stage of the history. Uniform stretch in plane strain tension to a strain, $\varepsilon_{11} = \varepsilon_T$, is followed by plane strain bending with no further overall stretch. That is, for $0 < \varepsilon_{11}^0 \le \varepsilon_T$, $\kappa = 0$ and $\varepsilon_{11} = \varepsilon_{11}^0$, while, subsequently, the middle surface strain is fixed at $\varepsilon_{11}^0 = \varepsilon_T$ and $\dot{\varepsilon}_{11} = \dot{\kappa} x_2$ with $\dot{\kappa} > 0$.

For the rate-independent *non-incremental theory*, the first increment following the onset of bending, minimum principle I is still given by (3.6) and (3.7), except that there is no constraint on the plastic strain-rate at the surfaces. Principle I says that the plastic strain-rate distribution must be uniform. Application of principle II then says that the amplitude of this uniform plastic strain-rate distribution must be zero. Thus, according to this theory, $\dot{\varepsilon}_{11}^P(x_2) = 0$ at the onset of bending. (This is true also for the stretch-passivation problem at $R = R_C$.) Predictions based on the rate-dependent version of the theory in Fig. 5a are consistent with the behavior described above. In the example shown, the layer is stretched well into the plastic range ($\sigma_{11} / \sigma_Y = 2$, $\varepsilon_T / \varepsilon_Y = 7.32$) and then subject to bending. The slope of the moment-curvature relation governing elastic incremental behavior, $\dot{M} / (Eh^3 \dot{\kappa}) = (8/9)$, is shown in Fig. 5a. The early stage of the bending response is nearly elastic and relatively insensitive to the values of the strain-rate sensitivity exponent chosen. After the onset of bending, there is no elastic gap but additional plasticity develops slowly.

For the *incremental theory*, the boundary value problem (3.4) for the first increment following the imposition of bending can be solved analytically with the result:

$$\frac{\dot{\varepsilon}_{11}^P}{\dot{\kappa} h} = K\left(\frac{x_2}{h} - \frac{\sinh(\beta x_2 / \ell)}{(\beta h / \ell)\cosh(\beta h / \ell)}\right), \quad x_2 \ge 0; \qquad \frac{\dot{\varepsilon}_{11}^P}{\dot{\kappa} h} = 0, \quad x_2 < 0 \tag{3.9}$$

$$\frac{\dot{M}}{Eh^3 \dot{\kappa}} = \frac{\dot{M} / \sigma_Y h^2}{\dot{\kappa} h / \varepsilon_Y} = \frac{8}{9}\left\{1 - \frac{K}{2}\left[1 - \frac{3}{(\beta h / \ell)^2}\left(1 - \frac{\tanh(\beta h / \ell)}{(\beta h / \ell)}\right)\right]\right\} \tag{3.10}$$

where K and β are given in (3.8). The limit $\ell \to 0$, $\dot{M} / (Eh^3\dot{\kappa}) = (8/9)\left(1 - K/2\right)$, applies to a layer of material without dependence on strain gradients. This initial slope and the elastic slope, $\dot{M} / (Eh^3\dot{\kappa}) = (8/9)$, are included with the full moment-curvature response following the onset of bending in Fig. 5b. The full response is generated by solving the minimum principle (3.4) sequentially, increment by increment. No reversed plastic straining occurs on the compressive side of the layer over the range of curvature imposed in Fig. 5b, $0 \le \kappa h / \varepsilon_Y \le 1$. The bending moment increases almost linearly over this range and is in close agreement with (3.10).

The incremental theory predicts that the moment-curvature relation following initial uniform stretch is increased above the classical plasticity prediction ($\ell = 0$), depending on ℓ / h. The response is significantly reduced below the initial elastic response predicted by the non-incremental theory. From a physical stand point, there is a significant difference between the predictions from the two types of theories for both this stretch-bend problem and the earlier passivation problem.

4. Detailed analysis of the re-emergence of plastic strain following passivation for the stretch problem for the formulation based on non-incremental stresses

As revealed in Section 3.1, the non-incremental theory suggests that plastic flow is interrupted when a layer which has been stretched uniformly into the plastic range to a stress σ_{11}^T experiences surface passivation with subsequent plastic straining blocked at its surfaces. Following passivation, the layer undergoes uniform incremental elastic behavior until plastic straining resumes at σ_{11}^C. Figure 1a presents the dependence of $R_C = \sigma_{11}^C / \sigma_{11}^T$ on ℓ / h based on the solution to the problem posed by (3.7). The resumption of plastic flow after the gap of elastic deformation gives rise to some challenging and interesting mathematical issues which will be addressed in this section.

The starting point is the solution to the eigenvalue problem (3.7), which is valid in this form as long as σ_{11} remains uniform. Denote the integrand of (3.7) by $f(y', y)$ with dependence on ℓ / h and R implicit. Because the integrand has no explicit x_2-dependence, a first integral of the Euler-Lagrange equation is $f - y'\partial f / \partial y' = c$. By symmetry, $y'(0) = 0$, and because the equation is homogeneous, one can require $y(0) = 1$, such that the first integral is

$$\frac{y^2}{\sqrt{((\ell/h)y')^2+y^2}}=1-R+Ry \tag{4.1}$$

The solution to (4.1) can be expressed in the form

$$\frac{x_2}{\ell}=\int_{y(x_2)}^{1}\frac{\left[1-R(1-y)\right]dy}{y\left\{y^2-\left[1-R(1-y)\right]^2\right\}^{1/2}} \tag{4.2}$$

or, with the variable transformation

$$\frac{1-R(1-y)}{y}=\cos\theta \qquad \Leftrightarrow \qquad y=\frac{R-1}{R-\cos\theta}, \tag{4.3}$$

in the form

$$\frac{x_2}{\ell}=\int_0^{\theta(x_2)}\frac{\cos\theta d\theta}{R-\cos\theta}=\frac{2R}{\left(R^2-1\right)^{1/2}}\tan^{-1}\left[\left(\frac{R+1}{R-1}\right)^{1/2}\tan\left(\frac{\theta}{2}\right)\right]-\theta \tag{4.4}$$

The largest possible value of x_2 is achieved when $\theta(x_2)=\pi/2$ for which the corresponding value of $y(x_2)$ is $y^*=(R-1)/R$. Thus,

$$\frac{h}{\ell}\leq\frac{2R}{\left(R^2-1\right)^{1/2}}\tan^{-1}\left[\left(\frac{R+1}{R-1}\right)^{1/2}\right]-\frac{\pi}{2} \tag{4.5a}$$

and the smallest value of R, $R=R_C$, for which this is true satisfies the equation

$$\frac{h}{\ell}=\frac{2R_C}{\left({R_C}^2-1\right)^{1/2}}\tan^{-1}\left[\left(\frac{R_C+1}{R_C-1}\right)^{1/2}\right]-\frac{\pi}{2} \tag{4.5b}$$

Equations (4.3) (4.4) and (4.5b) provide the plots of Fig. 1.

To facilitate discussion of the problem after the resumption of plastic flow, it is convenient to define $\overline{R}=\overline{\sigma}_{11}/\sigma_{11}^T$. The non-zero components of the Cauchy stress are σ_{11} and $\sigma_{33}=\sigma_{11}/2$, and the non-zero components of plastic strain are ε_{11}^P and $\varepsilon_{22}^P = -\varepsilon_{11}^P$, all functions of x_2 only, whereas ε_{11} is uniform and prescribed. Then,

$$\sigma_{11} = \frac{4}{3} E\left(\varepsilon_{11} - \varepsilon_{11}^P\right). \tag{4.6}$$

During plastic flow,

$$q_{11}^{UR} = \frac{2}{3}\sigma_0(\mathrm{E}_P)\frac{\dot{\varepsilon}_{11}^P}{\dot{\mathrm{E}}_P} \qquad \text{and} \qquad \tau_{112}^{UR} = \frac{2}{3}\ell^2\sigma_0(\mathrm{E}_P)\frac{\dot{\varepsilon}_{11.2}^P}{\dot{\mathrm{E}}_P} \tag{4.7}$$

where, expressed in terms of $\dot{\varepsilon}_{11}^P$,

$$\dot{\mathrm{E}}_P = \frac{2}{\sqrt{3}}\left[\left(\dot{\varepsilon}_{11}^P\right)^2 + \ell^2\left(\dot{\varepsilon}_{11,2}^P\right)^2\right]^{1/2}. \tag{4.8}$$

With

$$\Sigma^{UR} = \sqrt{3}[(q_{11}^{UR})^2 + (\tau_{112}^{UR})^2 / \ell^2]^{1/2}, \tag{4.9}$$

plastic flow does not occur when $\Sigma^{UR} < \sigma_0\left(\mathrm{E}_P\right)$. Since $q_{ij}^R = \tau_{ijk}^R = 0$ in this example, $\dot{\varepsilon}_{11}^P$ is determined from the equilibrium equation

$$q_{11}^{UR} - \tau_{112.2}^{UR} = s_{11} \equiv \frac{1}{2}\sigma_{11} = (2/3)E(\varepsilon_{11} - \varepsilon_{11}^P). \tag{4.10}$$

Strictly speaking, although study of minimum principle I *suggests* that plastic flow will resume as $\bar{R}$ passes through R_C, it only permits the firm conclusion that $\dot{\varepsilon}_{ij}^P = 0$ at $\bar{R} = R_C$ and does not help in continuing the solution beyond R_C. The remainder of this section is devoted to a resolution of this dilemma.

It is assumed that ε_{11} is prescribed as a monotone increasing function of time. Since rate-independent behavior is considered, ε_{11} itself can be taken as the time-like variable; passivation commenced at ε_{11}^T and plastic flow resumes at ε_{11}^C. *Henceforth, the suffixes 11 and 112 will be dropped*.

4.1 Direct derivation of R_C

Consider first the range $\varepsilon^T < \varepsilon < \varepsilon^C$ (the latter to be determined). By hypothesis, no plastic deformation has occurred since passivation so ε^P remains at the value ε^{PT}, and the stress σ^T corresponding to strain ε^T has the value $\sigma^T = (2/\sqrt{3})\sigma_0^T$ where $\sigma_0^T = \sigma_0(2\varepsilon^{PT}/\sqrt{3})$. The Cauchy stress σ exceeds σ^T but still the yield criterion is not met. Thus, it must be possible to construct (q^{UR}, τ^{UR}) satisfying equation (4.10), for which $\Sigma^{UR} < (\sqrt{3}/2)\sigma^T$. We now demonstrate that this is the case. Let

$$q^{UR} = \rho\cos\theta, \quad \tau^{UR}/\ell = -\rho\sin\theta, \tag{4.11}$$

where ρ is a constant and θ depends on x_2. The yield criterion will not be violated so long as

$$\rho < \sigma^T/2. \tag{4.12}$$

Substituting expressions (4.11) into (4.10) gives

$$\rho\cos\theta(1+\ell\,\theta') = \sigma/2 \tag{4.13}$$

with solution

$$x_2 = \ell\int_0^\theta \frac{\cos u\,du}{\sigma/(2\rho)-\cos u} \tag{4.14}$$

(choosing the constant so that θ is an odd function of x_2). Note that this integral is identical to the one developed in (4.4) with R replaced by $\hat{R} = \sigma/(2\rho)$. Reasoning similar to that following (4.4) implies that the solution is defined for all x_2 provided $R < R_C$, as defined in (4.5b). This, together with inequality (4.12), implies

$$\frac{\sigma}{R_C} < 2\rho < \sigma^T \;, \tag{4.15}$$

and such values of ρ exist so long as $R < R_C$.

4.2 Solution beyond R_C

The system of equations comprising (4.5) and (4.8), together with the boundary conditions $\dot{\varepsilon}^P(\pm h)=0$ and initial condition $\varepsilon^P=\varepsilon^{PT}$ can be approached by discretizing the time-like variable ε into finite steps of magnitude $\Delta\varepsilon$. A scheme for doing this is outlined in Appendix A. The main point of this section is to investigate the first development of the plastic deformation close to the resumption of yield. This requires study of the first increment, $k=0$, as defined in the Appendix, where a variational principle for individual time steps is derived. This can be treated analytically because $\varepsilon_0^P=(\sqrt{3}/2)(\mathrm{E}_P)_0=\varepsilon^{PT}$ is independent of x_2 at $\varepsilon=\varepsilon^T$. The variational principle (A.8) with $k=0$ implies

$$\Phi_0-(\varepsilon_1^P)'\frac{\partial\Phi_0}{\partial(\varepsilon_1^P)'}=c \tag{4.16}$$

where ε_1^P is the plastic strain at the total strain level $\varepsilon_1=\varepsilon^C+\Delta\varepsilon$. The constant c is obtained below. It will be convenient to drop the reference to $k=0$ and to write

$$y(x_2)=\varepsilon_1^P(x_2)-\varepsilon_0^P\equiv\varepsilon_1^P(x_2)-\varepsilon^{PT}, \tag{4.17}$$

so that

$$Y=\sqrt{y^2+\ell^2(y')^2}. \tag{4.18}$$

Since the main interest is in the asymptotic response as $\Delta\varepsilon\to 0$, $(\sigma_0)_{\frac{1}{2}}$ will be approximated and replaced by

$$(\sigma_0)_{\frac{1}{2}}=\sigma_0^T+\left(\alpha/\sqrt{3}\right)Y\ , \tag{4.19}$$

where α denotes the rate of hardening $d\sigma_0/d\mathrm{E}_P$ evaluated at $\mathrm{E}_P^T=\left(2/\sqrt{3}\right)\varepsilon_0^P$.

Equation (4.16) can then be written in the form

$$\frac{\alpha}{6}(2y^2-Y^2)+\frac{\sigma_0^T y^2}{\sqrt{3}Y}+\frac{E}{6}\left(y^2-4(\varepsilon_C+\frac{1}{2}\Delta\varepsilon-\varepsilon^{PT})y\right)=c \tag{4.20}$$

and since, by symmetry, $y'(0)=0$,

$$c=\frac{\alpha y(0)^2}{6}+\frac{\sigma_0^T y(0)}{\sqrt{3}}+\frac{E}{6}\left(y(0)^2-4(\varepsilon_C+\frac{1}{2}\Delta\varepsilon-\varepsilon^{PT})y(0)\right). \tag{4.21}$$

The solution of the differential equation (4.18), with Y related to y via (4.20) and (4.21), and the boundary condition $y(h)=0$ now satisfied, can be expressed as

$$x_2=h-\ell\int_0^y \frac{du}{\sqrt{Y(u)^2-u^2}}. \tag{4.22}$$

The still-unknown constant $y_0\equiv y(0)$ follows from the requirement for consistency that

$$h=\ell\int_0^{y_0} \frac{du}{\sqrt{Y(u)^2-u^2}} \tag{4.23}$$

or, writing $\bar{u}=u/y_0$ and $\bar{Y}(\bar{u})=Y(u)/y_0$,

$$h/\ell=\int_0^1 \frac{d\bar{u}}{\sqrt{\bar{Y}(\bar{u})^2-\bar{u}^2}}. \tag{4.24}$$

The equations (4.20) and (4.21) for $\bar{Y}=Y/y_0$ as a function of $\bar{y}=y/y_0$ can be re-expressed as the cubic equation

$$X^3-aX-1=0 \tag{4.25}$$

by making the change of variable

$$X=\left[\frac{3\sigma^T\bar{y}^2}{\alpha y_0}\right]^{-1/3}\bar{Y} \tag{4.26}$$

with

$$a=\left(\frac{3\sigma^T}{\alpha y_0\bar{y}^4}\right)^{1/3}\left\{\left(\frac{\sigma^C+(2E/3)\Delta\varepsilon}{\sigma^T}\right)(1-\bar{y})-1\right.$$

$$+\frac{2\alpha y_0}{3\sigma^T}(\bar{y}^2-\frac{1}{2})-\frac{Ey_0}{3\sigma^T}(1-\bar{y}^2)\Big\} \tag{4.27}$$

Let $X = F(a)$ be the (unique) positive real solution of (4.26) such that

$$\bar{Y} = \left(\frac{3\sigma^T \bar{y}^2}{\alpha y_0}\right)^{1/3} F(a) \tag{4.28}$$

Numerical solution of the identity (4.24) for y_0 at selected values of $\Delta\varepsilon$ can be obtained by straightforward numerical iteration. At each value of $\bar{u}$, $\bar{Y}(\bar{u})$ is given by (4.28) with $F(a)$ obtained numerically. A convenient normalization uses y_0/ε_Y, $\Delta\varepsilon/\varepsilon_Y$, $R_C = \sigma^C/\sigma^T$ and σ^T/σ_Y such that $y_0\alpha/\sigma^T = (y_0/\varepsilon_Y)(\alpha/E)/(\sigma^T/\sigma_Y)$ and, by (2.22), $\alpha/E = pN\left(\varepsilon^{PT}/\varepsilon_Y\right)^{N-1}$. The other terms in (4.27) and (4.28) can be expressed similarly such that the problem is completely specified by the set of parameters: N, p, ℓ/h and σ^T/σ_Y. Results for a specific example are plotted in Fig. 6. It is seen that y_0/ε_Y varies quadratically for small $\Delta\varepsilon/\varepsilon_Y$ and then linearly at larger values. Plots of the distribution of the plastic strain increment, $y(x)$, normalized by $y_0 = y(0)$ are presented in Fig. 7. The asymptotic results in these figures are derived below.

Since plastic flow re-initiates at $\varepsilon = \varepsilon^C$, it follows that $y_0 \to 0$ as $\Delta\varepsilon \to 0$. This motivates consideration of equation (4.25) as $y_0 \to 0$. With a slight departure from earlier notation, let

$$R = \frac{\sigma^C + (2E/3)\Delta\varepsilon}{\sigma^T} \equiv R_C + \frac{1}{2}r, \tag{4.29}$$

where $r = (4E/(3\sigma^T))\Delta\varepsilon$. Also, define y^* so that $a = 0$ when $\bar{y} = y^*$. Then, for any $\bar{y} \in (0, y^* - \delta]$, $a \to +\infty$ as $y_0 \to 0$, and for any $\bar{y} \in [y^* + \delta, 1]$, $a \to -\infty$ as $y_0 \to 0$ for any fixed $\delta > 0$. Note that

$$y^* \to \frac{R-1}{R} \text{ as } y_0 \to 0 \tag{4.30}$$

and $R \to R_C$ as $\Delta\varepsilon \to 0$.

Since

$$F(a) \sim \begin{cases} a^{1/2} \ \text{ as } \ a \to +\infty \\ -1/a \ \text{ as } \ a \to -\infty, \end{cases} \tag{4.31}$$

it follows that, as $y_0 \to 0$,

$$\bar{Y} \sim \begin{cases} \left(\dfrac{3\sigma^T}{\alpha y_0}\right)^{1/2} [R(1-\bar{y})-1]^{1/2} \ \text{ for } \ 0 < \bar{y} \le y^* - \delta \\ \dfrac{\bar{y}^2}{R(1-\bar{y})} \ \text{ for } \ y^* + \delta \le \bar{y} \le 1 \end{cases} \tag{4.32}$$

Now from equation (4.24), necessarily,

$$h/\ell > \int_{y^*+\delta}^{1} \frac{d\bar{y}}{\sqrt{\bar{Y}^2 - \bar{y}^2}} \tag{4.33}$$

and so, letting $y_0 \to 0$,

$$h/\ell \ge \int_{y_*+\delta}^{1} \frac{[1-R(1-\bar{u})]d\bar{u}}{\bar{u}\{\bar{u}^2 - [1-R(1-\bar{u})]^2\}^{1/2}} \tag{4.34}$$

for any $\delta > 0$ (but $\delta < 1 - y^*$) and $\Delta\varepsilon$ sufficiently small. This inequality remains true when $\delta = 0$. Note that (4.2) contains the same integral, evaluated in (4.4). Thus, $R \ge R_C$.

4.3 Asymptotic solution for small $\Delta\varepsilon$

The asymptotic relation between y_0 and $\Delta\varepsilon$ as $\Delta\varepsilon \to 0$ can be obtained from the asymptotic approximations (4.32). By direct integration,

$$\int_0^{y^*} \frac{d\bar{u}}{\sqrt{\bar{Y}^2 - \bar{u}^2}} \sim \int_0^{y^*} \frac{d\bar{u}}{\bar{Y}} \sim \left(\frac{\alpha y_0}{3\sigma^T}\right)^{1/2} \frac{2(R-1)^{1/2}}{R}, \tag{4.35}$$

while, by expanding the right side of (4.34) with $R = R_C + r/2$,

$$\int_{y^*}^{1} \frac{d\bar{y}}{\sqrt{\bar{Y}^2 - \bar{y}^2}} \sim \frac{h}{l} - \left(\frac{h/l + \pi/2 + R_C}{R_C (R_C^2 - 1)} \right) \frac{r}{2}. \tag{4.36}$$

The two expressions above sum to the required value h/ℓ if

$$y_0 = \frac{(h/\ell + \pi/2 + R_C)^2}{(16\alpha/(3\sigma^T))(R_C - 1)(R_C^2 - 1)^2} r^2. \tag{4.37}$$

Fig. 6b shows an example plot of y_0/ε_Y against $\Delta\varepsilon/\varepsilon_Y$ computed from the exact form of $\bar{Y}$ compared with the asymptotic result (4.37). As has been noted earlier, and as seen in Fig. 6a, the relationship between these two quantities is essentially linear for $\Delta\varepsilon/\varepsilon_Y > 0.001$. However, for $\Delta\varepsilon/\varepsilon_Y < 0.0001$ the relationship approaches the quadratic dependence on $\Delta\varepsilon$ implied by (4.37). Note that the fact that the asymptotic result gives $y_0/\Delta\varepsilon \to 0$ as $\Delta\varepsilon \to 0$ provides the conclusion asserted earlier that $\dot{\varepsilon}^P(x_2) = 0$ at $\varepsilon = \varepsilon_C$. The remarkably small range of validity of the asymptotic result reflects the highly singular nature of the problem and the unusual character of the boundary layer discussed next.

An asymptotic relation is also obtained for $y(x_2)$ in the boundary layer near the surface. For $x_2/h \to 1$, $(\ell\bar{y}')^2 >> \bar{y}^2$ and, thus, $\ell\bar{y}' \cong -\bar{Y}(0)$. By (4.27) and (4.31), with terms of order y_0 neglected, $\bar{Y}(0) \cong \left((3\sigma^T/\alpha y_0)(R_C - 1)\right)^{1/2}$. Thus, in the boundary layer near $x_2 = h$,

$$\bar{y}' \cong -\frac{1}{\ell}\sqrt{\frac{3\sigma^T}{\alpha y_0}(R_C - 1)} \quad \text{and} \quad \frac{y}{y_0} \cong \frac{1}{\ell}\sqrt{\frac{3\sigma^T}{\alpha y_0}(R_C - 1)}\,(h - x_2)$$

(4.38)

The asymptotic results for y/y_0 in Fig. 7b in the boundary layer have been computed with the above equation using the values of y_0 from the exact numerical scheme and thus they are not restricted to the small range of validity noted in connection with (4.37). The width of the boundary layer scales with $1/(\ell\sqrt{y_0})$. Thus, the width starts from zero in the limit when $\Delta\varepsilon = 0$ and increases as $\Delta\varepsilon$ increases, as seen in Fig. 7b. The strain-dependent width of the boundary gives rise to the singular behavior of the solution associated with resumption of plastic flow.

This problem also illustrates limitations of the non-incremental formulation with regard to determination of the stress quantities $q = q_{11}$ and $\tau = \tau_{112}$. During uniform plastic stretch prior to passivation, $q = s_{11}$ and $\tau = 0$. In the elastic gap period following passivation, q and τ cannot be determined by the theory. However, immediately after the resumption of plastic flow the distributions of q and τ are determined. In the boundary layer in the first increment of resumed plastic flow, $|\ell y'| >> y$ and, by (4.7), $q \cong 0$ and $\tau \cong \ell\sigma_0(\varepsilon_P^T)/\sqrt{3}$. Thus, only an infinitesimal increment of plastic flow is require to establish these stresses following the period in which they were undetermined.

4.4 An improved estimate for y_0

The reasoning as presented above provides convincing evidence that the plastic strain increment $\varepsilon^P - \varepsilon^{PT}$ is of order $(\Delta\varepsilon)^2$ as $\Delta\varepsilon \to 0$. If the variation were exactly quadratic, the central difference approximation that has been employed would be exact, and hence ensuring satisfaction of the governing equations at $\varepsilon_{\frac{1}{2}} = \varepsilon^C + \dfrac{1}{2}\Delta\varepsilon$ is appropriate. This requires, however, an expression for $\varepsilon^P_{\frac{1}{2}}$ which has so far been approximated as $\frac{1}{2}(\varepsilon^{PT} + \varepsilon_1^P)$, whereas the new approximation

$$\varepsilon^P_{\frac{1}{2}} = (3\varepsilon^{PT} + \varepsilon_1^P)/4 \tag{4.39}$$

is asymptotically exact. Similarly, $\dot{\mathrm{E}}_P \propto \Delta\varepsilon$ so that

$$(\mathrm{E}_P)_{\frac{1}{2}} = (\mathrm{E}_P)_0 + \frac{Y_0}{2\sqrt{3}} \tag{4.40}$$

is asymptotically exact. Adopting these expressions implies the replacement of α by $\alpha^* = \frac{1}{2}\alpha$. The stress $s_{\frac{1}{2}} = \sigma_{\frac{1}{2}}/2$ now becomes

$$s_{\frac{1}{2}} = (2E/3)\left(\varepsilon^C + \tfrac{1}{2}\Delta\varepsilon - \varepsilon^{PT} - (\varepsilon_1^P - \varepsilon^{PT})/4\right)$$

$$= \frac{\partial}{\partial \varepsilon_1^P} \left(\frac{1}{2} R\sigma^T (\varepsilon_1^P - \varepsilon^{PT}) - (E/12)(\varepsilon_1^P - \varepsilon^{PT})^2 \right). \tag{4.41}$$

While this modifies the full equation for y_0, the only effect on the asymptotic result (4.37) is to replace α by $\alpha^* = \alpha/2$, thus doubling the coefficient of r^2.

5. Summary: Implications of the examples of non-proportional loading

As noted in the Introduction, applications of strain gradient plasticity to problems with proportional, or nearly-proportional, loading are not problematic. For such applications even a deformation theory will generally give predictions that are similar to those of a genuine plasticity theory. The class of constitutive laws with non-incremental stresses proposed by Gudmundson (2004) and Gurtin and Anand (2005, 2009) was specifically constructed to be applicable to non-proportional loading problems because it is under these conditions that violations of the constraint on plastic dissipation will generally arise. The examples in this paper reveal that this construction gives rise to unanticipated mathematical and physical consequences. By contrast, the incremental theory generates mathematical problems and predictions which are less exceptional, mathematically and physically, and the predictions do not diverge in an unexpected manner from those widely explored for non-proportional loading problems with the context of conventional theories.

For the stretch-passivation problem, the non-incremental theory predicts a substantial "elastic gap" following passivation with no plastic straining. The extent of the gap depends on the material length parameter. Within the elastic gap the stress quantities q_{11} and τ_{112} are undetermined. As laid out in Section 4, the problem for the additional plastic strain following the resumption of plastic flow is a non-standard incremental problem that is inherently nonlinear. No elastic gap is predicted for the incremental theory, and the incremental relation between the average stress and stretch increment after passivation deviates from conventional elastic-plastic behavior in a continuous manner that depends on the amplitude of the material length parameter. Unlike the non-incremental theory, the stress quantities q_{11} and τ_{112} are well defined throughout the history and vary continuously with stretch.

The moment-curvature behaviors predicted by the two theories following the onset of bending in the stretch-bend problem are also markedly different. For the non-incremental theory, there is a substantial range of curvature in which the moment-curvature response is nearly elastic. Within this same curvature range, the prediction based on the incremental theory indicates that the moment-curvature behavior is significantly less stiff and approaches that from conventional plasticity theory as the material length parameter becomes small.

Several interesting and unusual mathematical problems based on the non-incremental theory for resumption of plastic flow following passivation have been analyzed. Minimum Principle I of Fleck and Willis (2009) leads to a non-linear eigenvalue problem with no acceptable solution. Problematically, it is not possible to impose the desired boundary condition that the plastic strain increment vanishes at the surfaces. This is traced to the feature that Minimum Principle I has the character of a forward Euler scheme, which is adequate for the analysis of continued plastic flow. To deliver the correct asymptotic behavior, it was essential to employ an incremental scheme that samples at the end of the load step. Following resumption of plastic flow, the solution has a steep boundary layer adjacent to the passivated surfaces, and the boundary layer width increases from zero.

For the non-incremental theory, the problem for the initial plastic yield stress, σ_C, of a layer passivated from the start and subject to a plane strain stretch displays the same behavior, with yield initiating at $R_C = \sigma_C / \sigma_Y$. The same feature arises for shearing of a layer with constrained plastic flow at its surfaces. Yield initiates at a stress level $\tau_C > \tau_Y$ where τ_Y is the initial yield stress in the absence of gradients. The delay in yielding in the shear problem was computed as a function of the dimensionless material length parameter by Niordson and Legarth (2010) using a rate-dependent version of the non-incremental theory. The computed ratio, τ_C / τ_Y, in Fig. 3a of Niordson and Legarth (2010) agrees with the results for R_C in Fig. 1 to within several percent when account is taken for different definitions of the material length parameter. Nielsen and Niordson (2014) have presented further results, including for rate-independent behavior. Their finite element discretization employed Minimum Principles I and II of Fleck and Willis (2009). They could not capture details of the very steep boundary layer in the early stages of plastic flow as their algorithm was initiated by a small elastic step, but otherwise their numerical method produces satisfactory predictions of overall shear stress-strain behavior and of shear strain distributions beyond the early stage of plastic flow.

A third class of strain gradient plasticity theories has been proposed in the literature (e.g., Bayley et al. 2006; Kuroda & Tvergaard 2010) which has not been considered in this paper. In this third class of theories the governing equations are postulated in weak form. It is not necessary to define additional stress quantities, such as $\mathbf{q}$ and $\boldsymbol{\tau}$, although, in principle, they could be identified. This class of theories is intrinsically incremental. Thus, we conjecture that their application to the stretch-passivation and stretch-bend problems would generate results similar to those predicted by the incremental theory, but we have not carried out the requisite calculations.

From a physical standpoint, there are significant differences between the predictions of the two classes of theories considered here for the stretch-passivation and the stretch-bend problems. For both problems, the non-incremental theory predicts an initial response that is either elastic or nearly elastic, while the incremental theory predicts an initial response that is much less stiff due to continued plastic flow. The difference is most marked for the stretch-passivation problem where for the non-incremental theory it can be noted from Fig. 1 that a moderate value of the material length parameter, $\ell / h = 0.5$, predicts an elastic gap having almost a 40% increase in stress before resumption of plastic flow following passivation. The incremental theory predicts plastic flow is not interrupted by passivation, only constrained giving rise to an increase in effective incremental stiffness. This clear difference in predictions suggests critical experiments to clarify the physical relevance of the two theories.

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Appendix: Discretization in the time-like variable

Define $\varepsilon_k = \varepsilon^C + k\Delta\varepsilon$, and let ε_k^P denote ε^P at load level ε_k. Assuming that ε_k^P has already been found, the problem is to find ε_{k+1}^P. For this purpose, note that $(\varepsilon_{k+1}^P - \varepsilon_k^P)/\Delta\varepsilon$ gives exactly $\dot{\varepsilon}^P$, at some value of ε between ε_k and ε_{k+1}. However, the form of the resulting differential equation for ε_{k+1}^P depends on exactly what the finite difference is taken to represent. The simplest assumption is to employ the forward difference approximation, that the finite difference delivers $\dot{\varepsilon}_k^P$, i.e., the derivative evaluated at ε_k. This, however, is of no use at the first step, $k=0$, because it will give the result already found from minimum principle I, i.e., $\dot{\varepsilon}_0^P = 0$. A better assumption would be to make the backward difference approximation to deliver $\dot{\varepsilon}_{k+1}$, which amounts to employing an implicit scheme for solving the system. However, both the forward difference and the backward difference approximations have an error of order $\Delta\varepsilon$, whereas the central difference approximation

$$(\varepsilon_{k+1}^P - \varepsilon_k^P)/\Delta\varepsilon \approx \dot{\varepsilon}^P_{k+\frac{1}{2}} \tag{A.1}$$

has an error of order $(\Delta\varepsilon)^2$. The use of this approximation is now pursued. It implies that (4.7) and (4.10) are satisfied at $\varepsilon_{k+\frac{1}{2}}$. Equation (4.10) thus requires an expression for $\varepsilon^P_{k+\frac{1}{2}}$. The most natural choice is

$$\varepsilon^P_{k+\frac{1}{2}} \approx \frac{1}{2}(\varepsilon_k^P + \varepsilon^P_{k+\frac{1}{2}}). \tag{A.2}$$

Equations (4.7) require $(\mathrm{E}_P)_{k+\frac{1}{2}}$. The only simple choice is to assume that E_P varies linearly on the interval $(\varepsilon_k, \varepsilon_{k+1})$, which gives

$$(\mathrm{E}_P)_{k+\frac{1}{2}} \approx \frac{1}{2}[(\mathrm{E}_P)_k + (\mathrm{E}_P)_{k+1}] = (\mathrm{E}_P)_k + \frac{1}{2}\Delta\varepsilon(\dot{\mathrm{E}}_P)_{k+\frac{1}{2}} \tag{A.3}$$

where $(\dot{\mathrm{E}}_P)_{k+\frac{1}{2}}$ is obtained from its exact formula (4.8) with $\dot{\varepsilon}^P_{k+\frac{1}{2}}$ given by (A.1).

With these approximations (now treated as though they are exact), the system that defines ε^P_{k+1} is

$$q^{UR}_{k+\frac{1}{2}} = \frac{1}{\sqrt{3}}\left(\sigma_0\right)_{k+\frac{1}{2}} \frac{\varepsilon^P_{k+1} - \varepsilon^P_k}{Y_k} \ , \ \text{and} \quad \tau^{UR}_{k+\frac{1}{2}} = \frac{1}{\sqrt{3}}\left(\sigma_0\right)_{k+\frac{1}{2}} \frac{(\varepsilon^P_{k+1})' - (\varepsilon^P_k)'}{Y_k} \tag{A.4}$$

where

$$\left(\sigma_0\right)_{k+\frac{1}{2}} = \sigma_0((\mathrm{E}_P)_k + (1/\sqrt{3})Y_k) \tag{A.5}$$

and

$$Y_k = \frac{\sqrt{3}}{2}\Delta\varepsilon(\dot{\mathrm{E}}_P)_{k+\frac{1}{2}} = \left\{[\varepsilon^P_{k+1} - \varepsilon^P_k]^2 + l^2[(\varepsilon^P_{k+1})' - (\varepsilon^P_k)']^2\right\}^{1/2} \tag{A.6}$$

along with the equilibrium equation

$$q^{UR}_{k+\frac{1}{2}} - (\tau^{UR}_{k+\frac{1}{2}})' = (2/3)E\left\{\varepsilon_{k+\frac{1}{2}} - \frac{1}{2}(\varepsilon^P_{k+1} + \varepsilon^P_k)\right\} \ . \tag{A.7}$$

It is worthwhile to note that this system is equivalent to the variational statement

$$\delta\int_{-h}^{h} \Phi_k(Y_k)dx_2 = 0 \ , \tag{A.8}$$

where

$$\Phi_k(Y_k) = \int_0^{Y_k} \left\{\frac{1}{\sqrt{3}}\sigma_0((\mathrm{E}_P)_k + (1/\sqrt{3})Z)\right\}dZ$$
$$+(1/6)E\left\{(\varepsilon^P_{k+1} + \varepsilon^P_k)^2 - 4\varepsilon_{k+\frac{1}{2}}(\varepsilon^P_{k+1} + \varepsilon^P_k)\right\} \tag{A.9}$$

The variation is with respect to ε^P_{k+1} and this minimum principle delivers ε^P_{k+1} as a function of the current state ε^P_k and the increment $\Delta\varepsilon$.

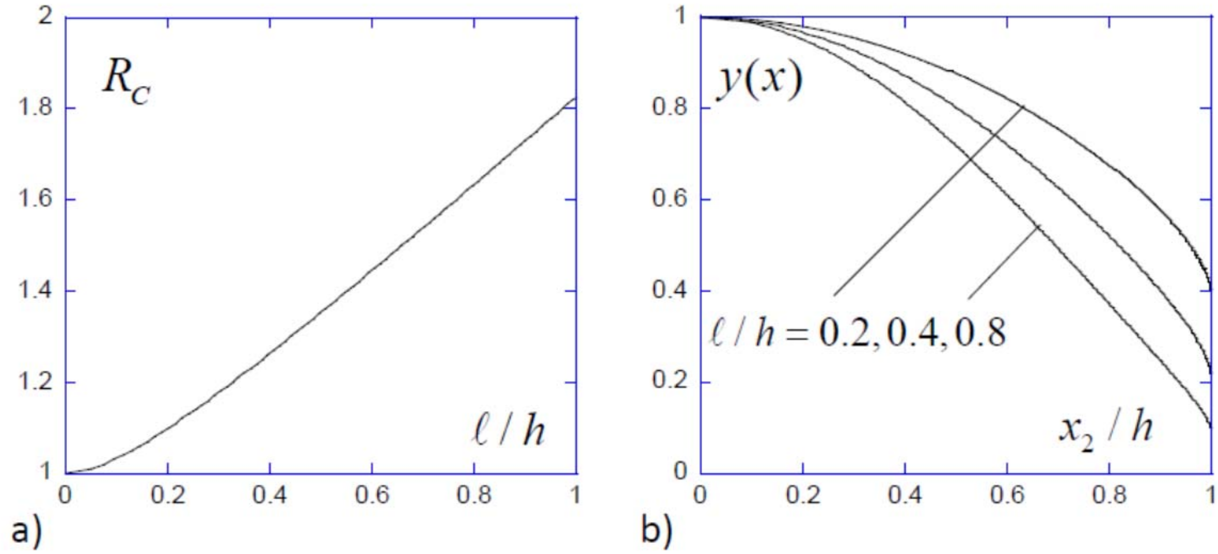

Figure 1. a) Plot of $R_C = \sigma_{11}^C / \sigma_{11}^T$ versus ℓ / h at resumption of plastic flow. b) Associated solutions y versus $x = x_2 / h$ for three values of ℓ / h.

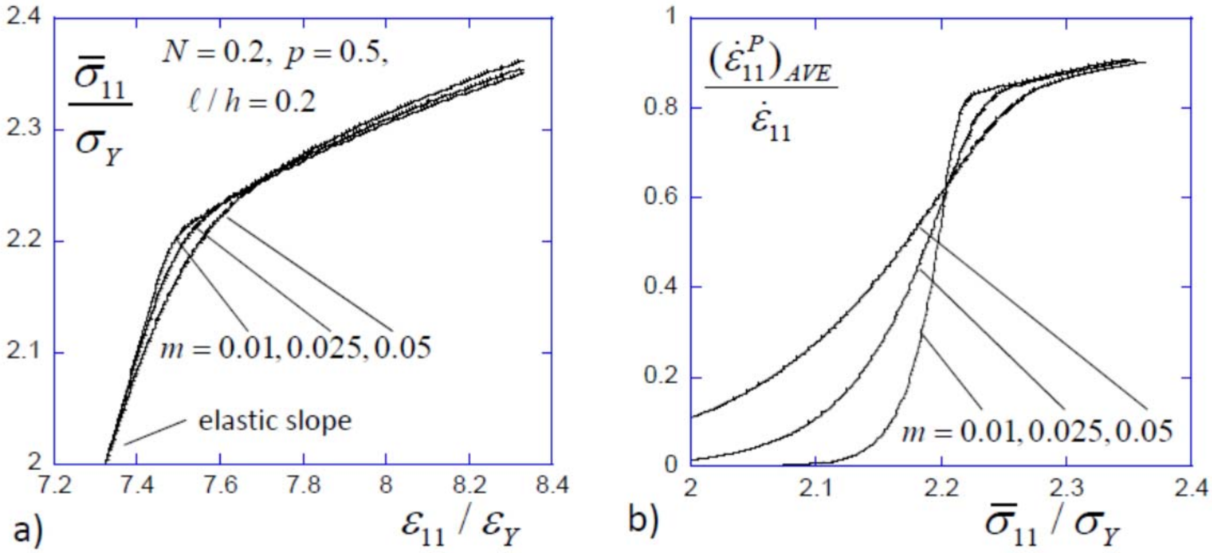

Figure 2. Rate-dependent predictions for the non-incremental theory showing (a) average stress and (b) normalized average plastic strain-rate following application of passivation at $\sigma_{11}^T / \sigma_Y = 2$ ($\varepsilon_{11} / \varepsilon_Y = 7.32$). The rate sensitivity exponent is m, $\ell / h = 0.2$, $N = 0.2$ and $p \equiv k\varepsilon_Y{}^N = 0.5$. For low rate sensitivity, i.e., $m = 0.01$, the stress at the resumption of plastic flow following passivation is nearly 10% above the stress at passivation in agreement with the eigenvalue prediction in Fig. 1a. The time lapsed in these simulations is $t\dot{\varepsilon}_R = 1$.

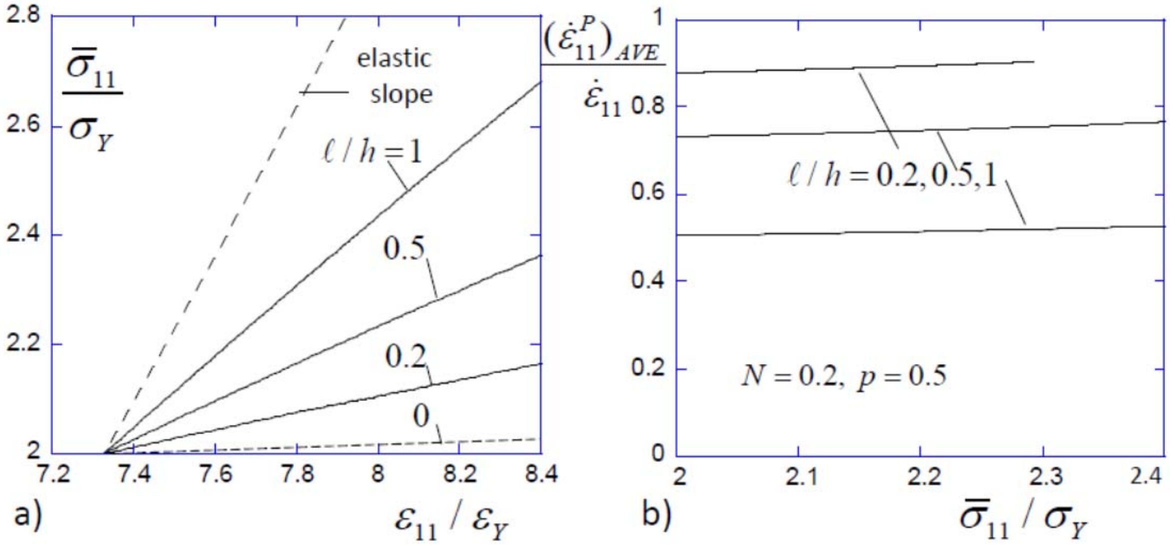

Figure 3. Rate-independent predictions for the incremental theory showing average stress in a) and normalized average plastic strain rate in b) following passivation at $\sigma_{11}^T / \sigma_Y = 2$ and $\varepsilon_{11}^T / \varepsilon_Y = 7.32$ for three values of ℓ / h, $N = 0.2$ and $p \equiv k \varepsilon_Y{}^N = 0.5$. In a), both the elastic slope and the slope in absence of any gradient effect, $\ell / h = 0$, are shown.

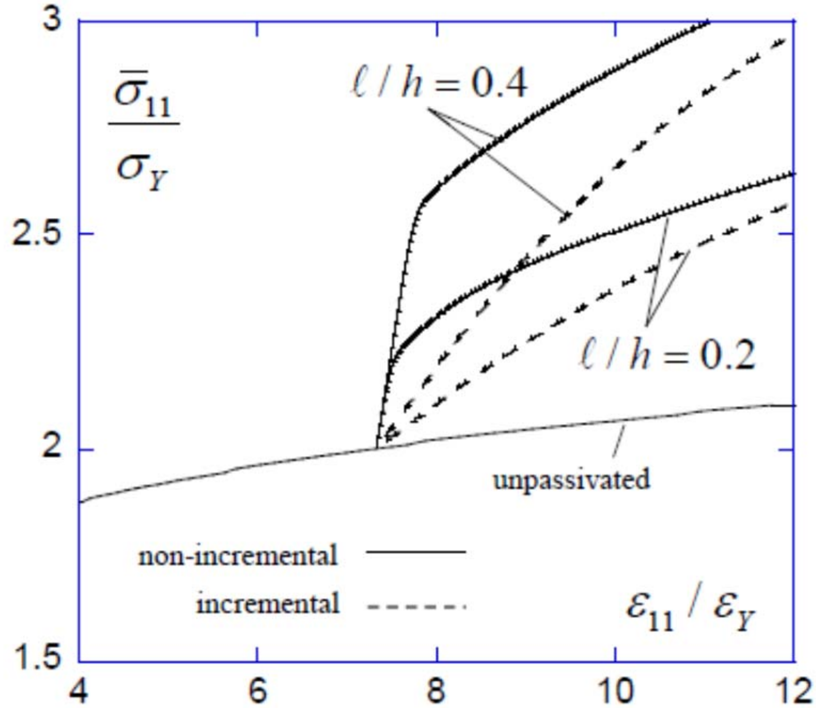

Figure 4. Comparison of the predictions of the two theories following application of passivation at $\sigma_{11}^T / \sigma_Y = 2$ ($\varepsilon_{11}^T / \varepsilon_Y = 7.32$) for $\ell / h = 0.2$ and 0.4, $N = 0.2$ and $p = k\varepsilon_Y{}^N = 0.5$. The results for the non-incremental theory were computed with $m = 0.025$.

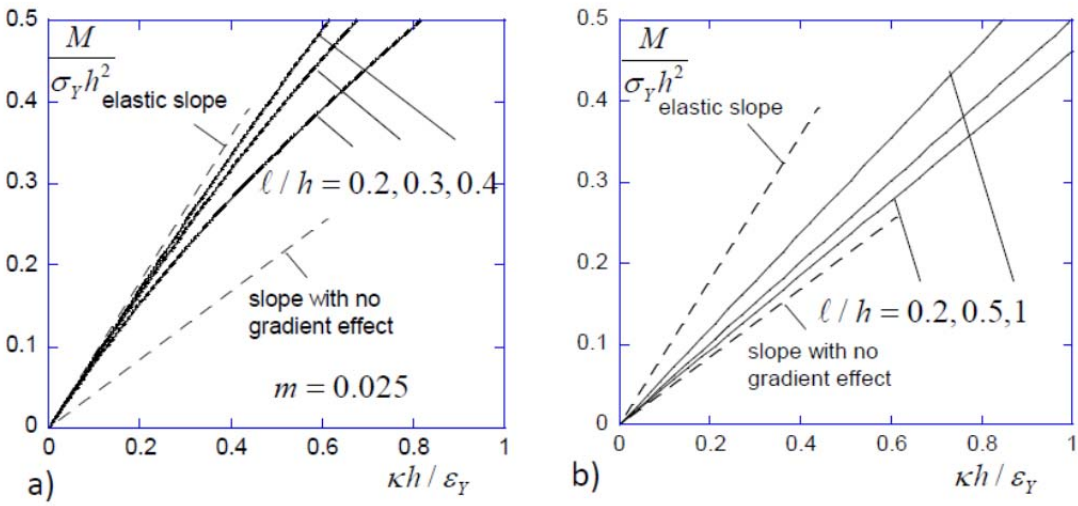

Figure 5. A layer in plane strain subjected to stretch followed by bending with no constraint on plastic flow at the surfaces. In this example, the layer is first stretched uniformly to $\sigma_{11} / \sigma_Y = 2$ and then subjected to bending with no further stretch, i.e., $\dot{\kappa} > 0$ with $\dot{\varepsilon}_{11}^0 = 0$. The moment-curvature response following the onset of bending: a) based on the non-incremental theory. b) based on the incremental theory. In these examples, $N = 0.2$, $p = k\varepsilon_Y{}^N = 0.5$. The rate-dependent simulations in a) have a rate-sensitivity index $m = 0.025$ and attain $\kappa h / \varepsilon_Y = 1$ at time $\dot{\varepsilon}_R t \cong 1$. The simulations in b) are rate independent.

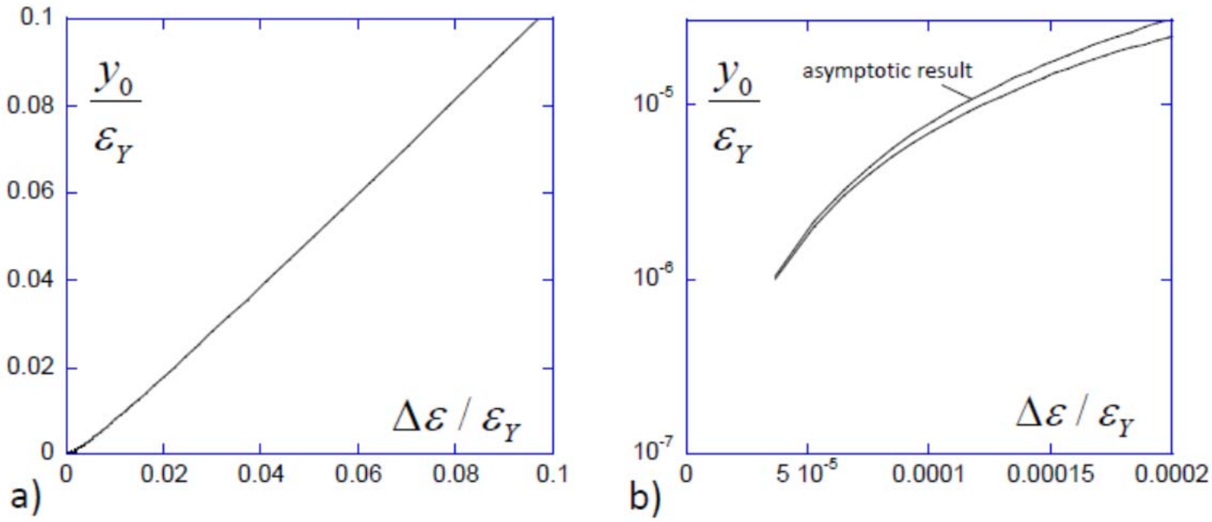

Figure 6. An example based on the non-incremental theory illustrating the relation between the amplitude of the normalized plastic strain increment at the center of the layer, y_0 / ε_Y, as a function of the normalized overall strain increment at the end of the first increment, $\Delta\varepsilon / \varepsilon_Y$, after plastic straining resumes following passivation. a) Nearly linear relationship except for very small first increments. b) Relationship for a very small first increment showing quadratic dependence on $\Delta\varepsilon / \varepsilon_Y$ approaching the asymptotic result (4.37). These results have been computed with $N = 0.2$, $p = 0.5$, $\ell / h = 0.5$ and $\sigma_{11}^T / \sigma_Y = 2$.

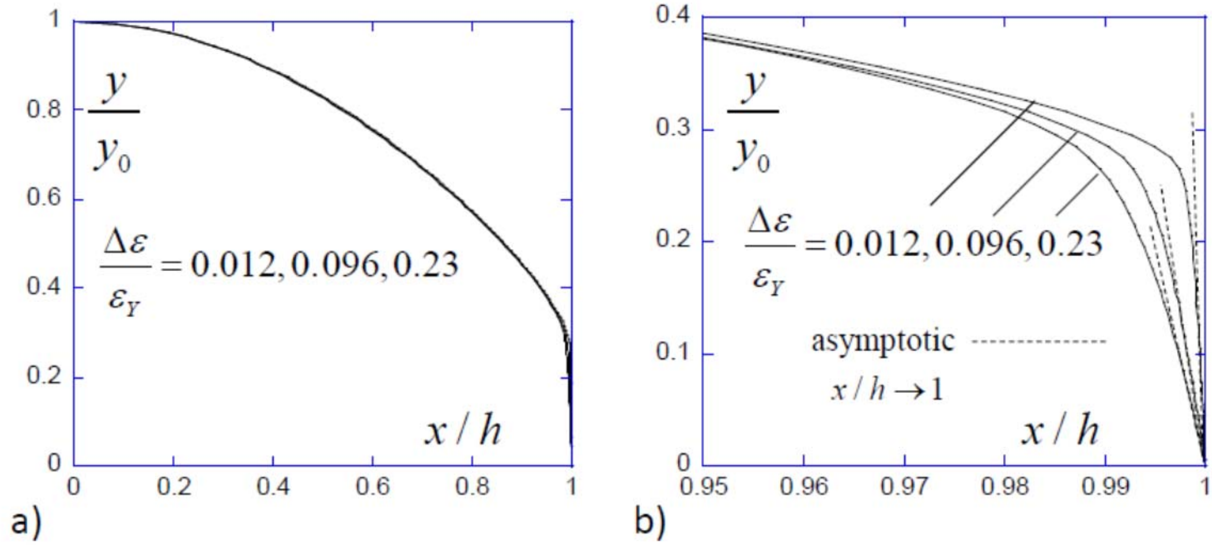

Figure 7. An example based on the non-incremental theory displaying the normalized distribution of the plastic strain at the end of the first increment of plastic straining following passivation for three values of prescribed $\Delta\varepsilon / \varepsilon_Y$. At the scale in a) the curves essentially overlay one another, but clear distinctions appear in the boundary layer in b). The steep boundary layer is captured by the asymptotic formula (4.38). These results have been computed with $N = 0.2$, $p = 0.5$, $\ell / h = 0.5$ and $\sigma_{11}^T / \sigma_Y = 2$. The relation between y_0 and $\Delta\varepsilon$ is plotted in Fig. 6.