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A PHENOMENOLOGICAL THEORY FOR STRAIN GRADIENT EFFECTS IN PLASTICITY

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ABSTRACT

A STRAIN GRADIENT THEORY of plasticity is introduced, based on the notion of statistically stored and geometrically necessary dislocations. The strain gradient theory fits within the general framework of couple stress theory and involves a single material length scale l. Minimum principles are developed for both deformation and flow theory versions of the theory which in the limit of vanishing l, reduce to their conventional counterparts: J_2 deformation and J_2 flow theory. The strain gradient theory is used to calculate the size effect associated with macroscopic strengthening due to a dilute concentration of bonded rigid particles; similarly, predictions are given for the effect of void size upon the macroscopic softening due to a dilute concentration of voids. Constitutive potentials are derived for this purpose.

1. INTRODUCTION

CONVENTIONAL CONSTITUTIVE theories of plasticity possess no material length scale. Predictions based on these theories involve only lengths associated with the geometry of the solid. For example, if one uses a conventional plasticity theory to predict the effect of well-bonded rigid particles on the flow stress of a metal matrix composite, the result will depend on the volume fraction, the shape and the spatial distribution of the particles, but not on their absolute size. There is accumulating experimental evidence for the existence of material size effects in plasticity, with the feature that the smaller the imposed geometric length scale relative to some material length scale, the stronger the material in its plastic response. Indentation tests show that inferred hardness increases with diminishing indent size for indents in the micron to sub-micron range (BROWN, 1993). In particulate reinforced metal matrix composites, small particles give rise to an enhanced rate of strain hardening compared to the same volume fraction of larger particles (KELLY and NICHOLSON, 1963; EBELING and ASHBY, 1966). Recent torsion tests on copper wires of diameter in the range 12–170 μm show that the thinner wires behave in a stronger manner than the thicker wires (FLECK *et al.*, 1993). A theory has been advanced by FLECK *et al.* (1993) for such phenomena based on the idea that a strain gradient leads to enhanced hardening due to the generation of geometrically necessary dislocations. Strain gradients exist in the region of an indent in an indentation test and near to the particles in a particulate composite. The smaller the indent or the smaller the particles, the larger the strain gradient and the larger the density of geometrically necessary dislocations, all other things being equal. In a torsion test the magnitude of the imposed strain gradient scales inversely with the wire diameter for a given level of surface shear strain. In each case the presence of a strain gradient leads to enhanced hardening. Conversely, strain gradients are absent in a simple tension test (prior to the onset of necking); FLECK *et al.* (1993) found that the uniaxial tensile response of the copper wires was independent of wire diameter.

It is now well established that the strain hardening of metals is due to the accumulation of dislocations. In a uniform strain field, dislocation storage is by random trapping and leads to the formation of dipoles. These dipoles act as a forest of sessile dislocations and strain hardening is associated with the elevation of the macroscopic flow stress required to cut the dipoles [see for example, HULL and BACON (1984)]. The randomly trapped dislocations are termed *statistically stored dislocations*. The von Mises effective plastic strain can be thought of as a useful scalar measure of their density in conventional plasticity theory. Gradients of plastic shear result in the storage of *geometrically necessary dislocations* (NYE, 1953; COTTRELL, 1964; ASHBY, 1970, 1971). A well-known example of this is in the plastic bending of a beam, where the plastic curvature κ of the beam can be considered to be due to the storage of extra half-planes of atoms, or, equivalently, to a uniform density of edge dislocations. The density ρ_G of these "geometrically necessary" edge dislocations is given by $|\kappa|/b$, where b is the magnitude of the Burgers vector of the dislocations. Note that $|\kappa|$ gives the magnitude of the strain gradient in the beam, and so ρ_G varies linearly with strain gradient.

FLECK *et al.* (1993) have developed a deformation theory version of plasticity which models the hardening due to both statistically stored and geometrically necessary dislocations. The degree of hardening due to statistically stored dislocations is assumed to scale with the von Mises effective strain. Hardening due to geometrically necessary dislocations is taken to scale with an isotropic scalar measure of the strain gradient in the deformed solid, and with a material length parameter l; this is made precise in Section 2 below. The theory fits neatly within the general framework of couple stress theory and reduces to conventional J_2 deformation theory in the absence of strain gradient effects, that is, when the geometric length scales are large compared to l. In both the previous paper by FLECK *et al.* (1993) and in the current paper finite strain effects are neglected: no distinction is made between the initial undeformed configuration and the current deformed configuration.

The outline of the paper is as follows. Couple stress theory is reviewed in order to introduce the stress and strain measures which are employed in the strain gradient theory. The deformation theory version of the strain gradient theory is outlined, and minimum principles are established for solving boundary value problems. A feature of the theory is the prediction of boundary layers near an interface or rigid boundary. The form of the boundary layer is explored and analytical expressions for it are given for the elastic solid. A J_2 flow theory version of the strain gradient theory follows naturally from the simpler deformation theory, and minimum principles are given in rate form. As in the case of their conventional counterparts, the deformation and flow theory versions give identical predictions when loading is proportional. Two examples are given where proportional loading is exhibited: macroscopic strengthening of a power law solid due to a dilute concentration of rigid spherical particles and softening due to a dilute concentration of spherical voids. The average macroscopic reponse is given in terms of a constitutive potential which makes use of the solution for an isolated inclusion (rigid particle or void) in an infinite solid. Detailed calculations are given for the isolated rigid particle and isolated void, and explicit predictions are presented on the effect of inclusion size. The results suggest that strain gradient effects have only a relatively minor influence on the softening due to voids and on their rate of growth, but large strengthening effects are predicted for rigid particles.

1.1. *Review of couple stress theory*

In couple stress theory it is assumed that a surface element dS of a body can transmit both a force vector $\mathbf{T}\,\mathrm{d}S$ (where $\mathbf{T}$ is the force traction vector) and a torque $\mathbf{q}\,\mathrm{d}S$ (where $\mathbf{q}$ is the couple stress traction vector). The surface forces are in equilibrium with the unsymmetric Cauchy stress, which is decomposed into a symmetric part $\boldsymbol{\sigma}$ and an anti-symmetric part $\boldsymbol{\tau}$. Now introduce the Cartesian coordinates x_i. Then $(\sigma_{ij}+\tau_{ij})$ denote the components of T_j on a plane with unit normal n_i such that

$$T_j = (\sigma_{ij}+\tau_{ij})n_i. \tag{1}$$

Similarly let μ_{ij} denote the components of q_j on a plane with normal n_i

$$q_j = \mu_{ij}n_i. \tag{2}$$

We refer to $\boldsymbol{\mu}$ as the couple stress tensor; it can be decomposed into a hydrostatic part $\mu\mathbf{I}$ (where $\mathbf{I}$ is the second order unit tensor) and a deviatoric component $\mathbf{m}$. KOITER (1964) has shown that the hydrostatic part of $\boldsymbol{\mu}$ does not enter the field equations and can legitimately be assumed to vanish; thus $\boldsymbol{\mu} \equiv \mathbf{m}$.

Equilibrium of forces within the body gives

$$\sigma_{ji,j}+\tau_{ji,j} = 0 \tag{3}$$

and equilibrium of moments gives

$$\tau_{jk} = -\tfrac{1}{2}e_{ijk}m_{pi,p}, \tag{4}$$

where we have neglected the presence of body forces and body couples. Thus $\boldsymbol{\tau}$ is specified once the distribution of $\mathbf{m}$ is known.

The principle of virtual work is conveniently formulated in terms of a virtual velocity field $\dot{u}_i$. The angular velocity vector $\dot{\theta}_i$ has the components

$$\dot{\theta}_i = \tfrac{1}{2}e_{ijk}\dot{u}_{k,j}. \tag{5}$$

Denoting the rate at which work is absorbed internally per unit volume by $\dot{U}$, the equation of virtual work reads

$$\int_V \dot{U}\,\mathrm{d}V = \int_S [T_i\dot{u}_i+q_i\dot{\theta}_i]\,\mathrm{d}S, \tag{6}$$

where the volume V is contained within the closed surface S. With the aid of the divergence theorem and the equilibrium relations (3) and (4), the right-hand side of

(6) may be rearranged to the form

$$\int_S [T_i \dot{u}_i + q_i \dot{\theta}_i]\, \mathrm{d}S = \int_V [\sigma_{ij}\dot{\varepsilon}_{ij} + m_{ij}\dot{\chi}_{ji}]\, \mathrm{d}V, \tag{7}$$

where the infinitesimal strain tensor is $\varepsilon_{ij} \equiv \frac{1}{2}(u_{i,j}+u_{j,i})$ and the infinitesimal curvature tensor is $\chi_{ij} \equiv \theta_{i,j}$. Note that the curvature tensor can be expressed in terms of the strain gradients as $\chi_{ij} = e_{ikl}\varepsilon_{jl,k}$. For the case of an incompressible solid, $\sigma_{ij}\dot{\varepsilon}_{ij} = s_{ij}\dot{\varepsilon}_{ij}$ where $\mathbf{s}$ is the deviatoric part of $\boldsymbol{\sigma}$.

2. DEFORMATION THEORY VERSION

FLECK *et al.* (1993) have developed a strain gradient version of J_2 deformation theory. In this section, we summarize their theory and give the associated minimum principles. A consequence of the higher order theory is the existence of a boundary layer at bimaterial interfaces. The nature of the boundary layer is revealed by considering the case of simple shear at a bimaterial interface.

The starting point in the deformation theory version of couple stress theory is to assume that the strain energy density w of a homogeneous isotropic solid depends upon the scalar invariants of the strain tensor $\boldsymbol{\varepsilon}$ and the curvature tensor $\boldsymbol{\chi}$. Since the rotation is defined as $\theta_i \equiv \frac{1}{2}e_{ijk}u_{k,j}$ [i.e. $\boldsymbol{\theta} \equiv \frac{1}{2}\,\mathrm{curl}\,(\mathbf{u})$], we have $\chi_{ii} = \frac{1}{2}e_{ijk}u_{k,ji} = 0$. Thus, $\boldsymbol{\chi}$ is an unsymmetric deviatoric tensor. We further assume the solid is incompressible and so the symmetric tensor $\boldsymbol{\varepsilon}$ is also deviatoric. The von Mises strain invariant $\varepsilon_e \equiv \sqrt{\frac{2}{3}\varepsilon_{ij}\varepsilon_{ij}}$ is used to represent the contribution to w from statistically stored dislocations and the invariant $\chi_e \equiv \sqrt{\frac{2}{3}\chi_{ij}\chi_{ij}}$ is used to represent the contribution to w from geometrically necessary dislocations. Any contribution to w from the invariant $\chi_{ij}\chi_{ji}$ is neglected for the sake of simplicity (though it could be included in an obvious and straightforward manner). It is mathematically convenient to assume that w depends only upon the single scalar measure $\mathscr{E}$ where

$$\mathscr{E}^2 \equiv \varepsilon_e^2 + l^2\chi_e^2. \tag{8}$$

Here, l is the material length scale introduced into the constitutive law, required on dimensional grounds. Following the arguments presented by FLECK *et al.* (1993), l may be interpreted loosely as the free slip distance between statistically stored dislocations. If we take the density ρ_S of statistically stored dislocations to be linear in ε_e and the density ρ_G of geometrically necessary dislocations to be linear in χ_e, then $\mathscr{E}$ may be interpreted as the harmonic mean of ρ_S and ρ_G, and is a useful measure of the total dislocation density.

Next define an overall stress measure Σ as the work conjugate of $\mathscr{E}$, with

$$\Sigma = \frac{\mathrm{d}w(\mathscr{E})}{\mathrm{d}\mathscr{E}} \tag{9}$$

and note that the overall stress Σ is a unique function of the overall strain measure $\mathscr{E}$. The work done on the solid per unit volume equals the increment in strain energy,

$$\delta w = s_{ij}\delta\varepsilon_{ij} + m_{ij}\delta\chi_{ji}, \tag{10}$$

which enables one to determine **s** and **m** in terms of the strain state of the solid as

$$s_{ij} = \frac{\partial w}{\partial \varepsilon_{ij}} = \Sigma \frac{\partial \mathscr{E}}{\partial \varepsilon_{ij}} = \frac{2}{3} \frac{\Sigma}{\mathscr{E}} \varepsilon_{ij} \tag{11a}$$

and

$$m_{ij} = \frac{\partial w}{\partial \chi_{ji}} = \Sigma \frac{\partial \mathscr{E}}{\partial \chi_{ji}} = \tfrac{2}{3} l^2 \frac{\Sigma}{\mathscr{E}} \chi_{ji}. \tag{11b}$$

Substitution of (11a) into the expression $\varepsilon_e^2 = \frac{2}{3}\varepsilon_{ij}\varepsilon_{ij}$ and (11b) into $\chi_e^2 = \frac{2}{3}\chi_{ij}\chi_{ij}$ gives, via (8),

$$\Sigma^2 = \sigma_e^2 + l^{-2} m_e^2, \tag{12}$$

where $\sigma_e \equiv \sqrt{\frac{3}{2} s_{ij} s_{ij}}$ is the usual von Mises effective stress and $m_e \equiv \sqrt{\frac{3}{2} m_{ij} m_{ij}}$ is the analogous effective couple stress. The measures σ_e and m_e are the work conjugates of ε_e and χ_e, respectively, such that $\mathrm{d}w = \sigma_e\,\mathrm{d}\varepsilon_e + m_e\,\mathrm{d}\chi_e$. Indeed, this work relation may be used as the defining equation for σ_e and m_e. The deformation theory is fully prescribed once a functional form is assumed for Σ in terms of $\mathscr{E}$. In Section 4 below we adopt the power law relation

$$\frac{\mathscr{E}}{\mathscr{E}_0} = \left(\frac{\Sigma}{\Sigma_0}\right)^n. \tag{13}$$

The constitutive description (8)–(12) may be derived in an alternative direct manner through the following formal mathematical device. In the absence of couple stresses the deviatoric part of the symmetric Cauchy stress tensor **s** may be represented by a five-dimensional vector. When couple stresses are present the role of **s** is replaced by that of the 13-dimensional vector $\boldsymbol{\Sigma} = (\mathbf{s}, l^{-1}\mathbf{m}^{\mathrm{T}})$, where $\mathbf{m}^{\mathrm{T}}$ is the transpose of $\mathbf{m}$; $\boldsymbol{\Sigma}$ is made up of the five symmetric components of **s** and the eight components of the unsymmetric, deviatoric tensor $l^{-1}\mathbf{m}^{\mathrm{T}}$. Similarly, when couple stresses are present the five-dimensional deviatoric strain measure $\boldsymbol{\varepsilon}' \equiv \boldsymbol{\varepsilon} - \frac{1}{3}\mathbf{I}\,\mathrm{tr}\,(\boldsymbol{\varepsilon})$ is replaced by the 13-dimensional vector $\boldsymbol{\mathscr{E}} \equiv (\boldsymbol{\varepsilon}', l\boldsymbol{\chi})$.

In the general case the solid is assumed to be *compressible* such that $\varepsilon_{kk} \neq 0$. The strain energy density w of the couple stress hyperelastic solid is taken to depend only upon the volumetric strain $\varepsilon_m \equiv \varepsilon_{kk}$ and the scalar invariant $\mathscr{E} = \sqrt{\frac{2}{3}\boldsymbol{\mathscr{E}} \cdot \boldsymbol{\mathscr{E}}}$. This definition of $\mathscr{E}$ is identical to that given in (8). The total differential of w may be written as

$$\delta w(\mathscr{E}, \varepsilon_m) = \Sigma \delta \mathscr{E} + \sigma_m \delta \varepsilon_m, \tag{14}$$

where $\Sigma \equiv \partial w/\partial \mathscr{E}$ is the overall effective stress and $\sigma_m \equiv \partial w/\partial \varepsilon_m = \frac{1}{3}\sigma_{kk}$ is the mean stress. Note that Σ is the work conjugate of $\mathscr{E}$, and σ_m is the work conjugate of ε_m. The work increment per unit volume is

$$\delta w(\mathscr{E}, \varepsilon_m) = \boldsymbol{\sigma} : \boldsymbol{\delta\varepsilon} + \mathbf{m}^{\mathrm{T}} : \boldsymbol{\delta\chi} = \mathbf{s} : \boldsymbol{\delta\varepsilon}' + \mathbf{m}^{\mathrm{T}} : \boldsymbol{\delta\chi} + \sigma_m \delta\varepsilon_m = \boldsymbol{\Sigma} \cdot \boldsymbol{\delta\mathscr{E}} + \sigma_m \delta \varepsilon_m. \tag{15}$$

The stress state corresponding to any given strain state follows directly from (14) and

(15) as

$$\mathbf{s} = \frac{\partial w}{\partial \boldsymbol{\varepsilon}'} = \frac{\partial w}{\partial \mathscr{E}}\frac{\partial \mathscr{E}}{\partial \boldsymbol{\varepsilon}'} = \frac{2}{3}\frac{\Sigma}{\mathscr{E}}\boldsymbol{\varepsilon}', \tag{16a}$$

$$l^{-1}\mathbf{m}^{\mathrm{T}} = \frac{\partial w}{l\partial \boldsymbol{\chi}} = \frac{\partial w}{\partial \mathscr{E}}\frac{\partial \mathscr{E}}{l\partial \boldsymbol{\chi}} = \frac{2}{3}\frac{\Sigma}{\mathscr{E}}l\boldsymbol{\chi} \tag{16b}$$

and

$$\sigma_{\mathrm{m}} = \frac{\partial w}{\partial \varepsilon_{\mathrm{m}}}. \tag{16c}$$

Substitution of (16a, b) for the components of $\mathscr{E} \equiv (\boldsymbol{\varepsilon}', l\boldsymbol{\chi})$ into $\mathscr{E} = \sqrt{\frac{2}{3}\mathscr{E}\cdot\mathscr{E}}$ results in the explicit expression (12) for the overall effective stress.

For the case of a linear elastic solid, the strain energy density w may be written as

$$w = \frac{E}{2(1+\nu)}\left(\frac{\nu}{1-2\nu}\varepsilon_{\mathrm{m}}^2 + \varepsilon_{ij}\varepsilon_{ij} + l^2\chi_{ij}\chi_{ij}\right), \tag{17}$$

where E is Young's modulus and ν is Poisson's ratio. The explicit dependence of w upon the invariants ε_{m} and $\mathscr{E}$ is indicated by rearranging (17) to the form

$$w = \tfrac{1}{2}K\varepsilon_{\mathrm{m}}^2 + \tfrac{3}{2}G\mathscr{E}^2, \tag{18}$$

where $K \equiv E/3(1-2\nu)$ is the bulk modulus and $G \equiv E/2(1+\nu)$ is the shear modulus of the solid.

In the case of an *incompressible* solid w depends only upon $\mathscr{E}$, and the term $\sigma_{\mathrm{m}}\delta\varepsilon_{\mathrm{m}}$ is dropped from the expression (15) for the work increment per unit volume; equations (15) and (16) then reduce to (10) and (11), respectively.

2.1. *Minimum principles*

KOITER (1964) has given a principle of minimum potential energy and a principle of minimum complementary energy for a linear elastic solid which supports couple stresses. These minimum principles can be extended straightforwardly in the small strain context for a non-linear elastic solid where the strain energy density w depends upon both $\boldsymbol{\varepsilon}$ and $\boldsymbol{\chi}$. Uniqueness of solution emerges naturally in the proofs of the minimum principles.

Consider a body of volume V and surface S comprised of a non-linear elastic solid; the solid satisfies the constitutive law given by the first part of equations (16a–c) (that is, $\sigma_{ij} = \partial w/\partial\varepsilon_{ij}$ and $m_{ij} = \partial w/\partial\chi_{ji}$). A stress traction T_i^0 and couple stress traction q_i^0 act on a portion S_{T} of the surface of the body. On the remaining portion S_u of the surface the displacement is prescribed as u_i^0 and the rotation is prescribed as θ_i^0. Then the following minimum principles may be stated.

Principle of minimum potential energy. Consider all admissible displacement fields u_i which satisfy $u_i = u_i^0$ and $\theta_i \equiv \frac{1}{2}e_{ijk}u_{k,j} = \theta_i^0$ on a part of the boundary S_u. Let $\varepsilon_{ij} \equiv \frac{1}{2}(u_{i,j}+u_{j,i})$ and $\chi_{ip} \equiv \frac{1}{2}e_{ijk}u_{k,jp}$ be the state of strain derived from u_i, and take $(\boldsymbol{\sigma}, \mathbf{m})$ to be the stress field associated with $(\boldsymbol{\varepsilon}, \boldsymbol{\chi})$ via $\sigma_{ij} = \partial w/\partial\varepsilon_{ij}$ and $m_{ij} = \partial w/\partial\chi_{ji}$.

Define the potential energy $P(\mathbf{u})$ as

$$P(\mathbf{u}) = \int_V w(\boldsymbol{\varepsilon}, \boldsymbol{\chi}) \, \mathrm{d}V - \int_{S_T} [T_i^0 u_i + q_i^0 \theta_i] \, \mathrm{d}S, \tag{19}$$

where the surface integral is taken over that part S_T of the surface of the body over which loading T_i and q_i are prescribed. Then the principle of minimum potential energy is:

Provided w *is strictly convex in* $\boldsymbol{\varepsilon}$ *and* $\boldsymbol{\chi}$ *the potential energy* P(**u**), *considered as a functional in the class of kinematically admissible displacement fields* **u**, *attains an absolute minimum for the actual displacement field.*

The uniqueness of solution is implied in the statement that the minimum is absolute.

Proof: Take **u** to represent the equilibrium solution; it is associated with the strain state $(\boldsymbol{\varepsilon}, \boldsymbol{\chi})$ and stress state $(\boldsymbol{\sigma}, \mathbf{m})$ via the constitutive law. The anti-symmetric part $\boldsymbol{\tau}$ of the Cauchy stress follows directly from the spatial gradient of **m** by (4). Let $\mathbf{u}+\Delta\mathbf{u}$ represent a kinematically admissible configuration associated with $(\boldsymbol{\varepsilon}+\Delta\boldsymbol{\varepsilon}, \boldsymbol{\chi}+\Delta\boldsymbol{\chi})$ and the non-equilibrium stress state $(\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma}, \mathbf{m}+\Delta\mathbf{m})$. The difference in potential energy of the body in the two states is $\Delta P = P(\mathbf{u}+\Delta\mathbf{u}) - P(\mathbf{u})$. Direct evaluation of ΔP using (19) and the virtual work statement (7) gives

$$\Delta P = \int_V \left[w(\boldsymbol{\varepsilon}+\Delta\boldsymbol{\varepsilon}, \boldsymbol{\chi}+\Delta\boldsymbol{\chi}) - w(\boldsymbol{\varepsilon}, \boldsymbol{\chi}) - \Delta\varepsilon_{ij} \frac{\partial w(\boldsymbol{\varepsilon}, \boldsymbol{\chi})}{\partial \varepsilon_{ij}} - \Delta\chi_{ij} \frac{\partial w(\boldsymbol{\varepsilon}, \boldsymbol{\chi})}{\partial \chi_{ij}} \right] \mathrm{d}V. \tag{20}$$

Provided $w(\boldsymbol{\varepsilon}, \boldsymbol{\chi})$ is strictly convex in $\boldsymbol{\varepsilon}$ and $\boldsymbol{\chi}$ the integrand of (20) is positive and the potential energy has a proper absolute minimum in the equilibrium configuration.

For the special case where w depends only upon $\mathscr{E}$ it may be shown that w is strictly convex in $\boldsymbol{\varepsilon}$ and $\boldsymbol{\chi}$ provided $\Sigma = \mathrm{d}w(\mathscr{E})/\mathrm{d}\mathscr{E}$ is an increasing function of $\mathscr{E}$. It is known from convex function theory that w is convex in $\boldsymbol{\varepsilon}$ and $\boldsymbol{\chi}$ provided (i) w is strictly convex in $\mathscr{E}$ and (ii) $\mathscr{E}$ is strictly convex in $\boldsymbol{\varepsilon}$ and $\boldsymbol{\chi}$ [see for example ROCKAFELLAR (1970), p. 32]. With the definition (8) $\mathscr{E}$ is strictly convex in $\boldsymbol{\varepsilon}$ and $\boldsymbol{\chi}$. Provided $\Sigma = \mathrm{d}w(\mathscr{E})/\mathrm{d}\mathscr{E}$ is an increasing function of $\mathscr{E}$ (that is, provided the tangent modulus $E_{\mathrm{T}} = \mathrm{d}\Sigma/\mathrm{d}\mathscr{E}$ is positive), w is strictly convex in $\mathscr{E}$. Thus, for the case where w depends only on $\mathscr{E}$, w is convex in $\boldsymbol{\varepsilon}$ and $\boldsymbol{\chi}$ provided we ensure that $\Sigma = \mathrm{d}w(\mathscr{E})/\mathrm{d}\mathscr{E}$ is an increasing function of $\mathscr{E}$. For such a solid, the potential energy (19) is minimized by the actual displacement field.

Principle of minimum complementary energy. In order to develop a minimum principle for the complementary energy it is necessary to introduce the stress potential $\phi(\boldsymbol{\sigma}, \mathbf{m})$ which is the dual of $w(\boldsymbol{\varepsilon}, \boldsymbol{\chi})$,

$$\phi(\boldsymbol{\sigma}, \mathbf{m}) = \int_0^{\boldsymbol{\sigma}} \varepsilon_{ij} \, \mathrm{d}\sigma_{ij} + \int_0^{\mathbf{m}} \chi_{ji} \, \mathrm{d}m_{ij} = \sigma_{ij}\varepsilon_{ij} + m_{ij}\chi_{ji} - w(\boldsymbol{\varepsilon}, \boldsymbol{\chi}). \tag{21}$$

Thus the strain state $(\boldsymbol{\varepsilon}, \boldsymbol{\chi})$ may be taken to be derived from the stress state $(\boldsymbol{\sigma}, \mathbf{m})$ via $\varepsilon_{ij} = \partial\phi/\partial\sigma_{ij}$ and $\chi_{ij} = \partial\phi/\partial m_{ji}$.

Define the complementary energy $C(\boldsymbol{\sigma}, \mathbf{m})$ as

$$C(\boldsymbol{\sigma}, \mathbf{m}) = \int_V \phi(\boldsymbol{\sigma}, \mathbf{m})\,\mathrm{d}V - \int_{S_u} [T_i u_i^0 + q_i \theta_i^0]\,\mathrm{d}S \tag{22}$$

and consider all admissible equilibrium stress fields $(\boldsymbol{\sigma}, \mathbf{m})$ which satisfy the traction boundary conditions $(\phi_{ij} + \tau_{ij})n_i = T_j^0$ and $m_{ij}n_i = q_j^0$ on S_T. Let u_i^0 and θ_i^0 be prescribed on the remaining portion S_u, and define $(\boldsymbol{\varepsilon}, \boldsymbol{\chi})$ as the state of strain associated with the stress $(\boldsymbol{\sigma}, \mathbf{m})$ via $\varepsilon_{ij} = \partial\phi/\partial\sigma_{ij}$ and $\chi_{ij} = \partial\phi/\partial m_{ji}$. Then, the principle of minimum complementary energy may be stated as:

Provided ϕ is strictly convex in $\boldsymbol{\sigma}$ and $\mathbf{m}$ the complementary energy $C(\boldsymbol{\sigma}, \mathbf{m})$, considered as a functional in the statically admissible stress fields $(\boldsymbol{\sigma}, \mathbf{m})$, attains an absolute minimum for the actual stress field.

Since the minimum is absolute, uniqueness of solution is assured.

The proof proceeds along similar lines to that given above for the principle of minimum potential energy, and is not given here. A full discussion for the linear case is given by KOITER (1964). For the actual solution $(\mathbf{u}, \boldsymbol{\sigma}, \mathbf{m})$ which gives rise to the minimum value of potential energy $P_{\min}$ and the minimum value of complementary energy $C_{\min}$, it is readily shown that

$$P_{\min} + C_{\min} = 0. \tag{23}$$

2.2. *Boundary layer near a bimaterial interface*

The presence of strain gradient dependence within the constitutive law leads to a higher order set of partial differential equations governing deformation of the body and to higher order boundary conditions. An extra contribution $q_i\dot{\theta}_i$ appears in the boundary term of the virtual work statement (7). The higher order boundary condition gives rise to the existence of a boundary layer adjacent to certain types of boundaries in the solid. We explore this phenomenon for the simple but instructive case of an interface between two elastic solids under remote simple shear, as shown in Fig. 1. This simple example will inform the behaviour at the interface between a bonded particle and the matrix in the particle reinforcement problem studied later.

It is assumed that both materials are incompressible and satisfy the constitutive description (8)–(12). Material 1 lies above the interface and possesses a shear modulus G_1, such that (13) may be re-expressed as $\Sigma = 3G_1\mathscr{E}$; the material length scale l in (8), (11b) and (12) is designated l_1 for material 1. Similarly, material 2 possesses a shear modulus G_2 and obeys $\Sigma = 3G_2\mathscr{E}$; it is ascribed a material length scale l_2.

The bimaterial is subjected to remote simple shear. In the Cartesian reference frame defined in Fig. 1, the only non-vanishing displacement component u_1 is taken to be a function solely of x_2. The non-vanishing components of the strain tensor and curvature tensor may be expressed in terms of the engineering shear strain $\gamma = u_{1,2}$ as

$$\varepsilon_{12} = \varepsilon_{21} = \tfrac{1}{2}\gamma \quad \text{and} \quad \chi_{32} = \theta_{3,2} = -\tfrac{1}{2}\gamma_{,2}. \tag{24}$$

The active stresses within the solid are $\tau_S \equiv \sigma_{21} + \tau_{21}$, $\tau_T \equiv \sigma_{12} + \tau_{12}$ and the couple stress $m \equiv m_{23}$; these stress components may vary with x_2 but not with x_1 or x_3.

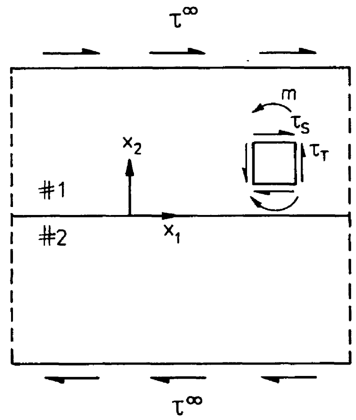

FIG. 1. A bimaterial interface under remote simple shear. For the couple stress solid, a boundary layer exists adjacent to the interface in both solids.

Force equilibrium (3) demands that τ_S equals the remote shear stress τ^{∞}. Moment equilibrium (4) gives

$$m_{,2} = \tau_S - \tau_T. \tag{25}$$

The constitutive law (8)–(13) reduces to

$$\tau_S + \tau_T = 2G_1\gamma \quad \text{and} \quad m = -l_1^2 G_1 \gamma_{,2} \tag{26a}$$

in material 1, and

$$\tau_S + \tau_T = 2G_2\gamma \quad \text{and} \quad m = -l_2^2 G_2 \gamma_{,2} \tag{26b}$$

in material 2. Elimination of τ_S, ε_T and m from (25) using (26a, b) gives the governing differential equation for γ as

$$-\tfrac{1}{2}l_1^2\gamma_{,22} + \gamma = \frac{\tau^{\infty}}{G_1} \tag{27a}$$

in material 1, and

$$-\tfrac{1}{2}l_2^2\gamma_{,22} + \gamma = \frac{\tau^{\infty}}{G_2} \tag{27b}$$

in material 2. In order to determine the unique solution to (27a) and (27b) we apply the following boundary conditions:

(I) as $|x_2| \to \infty$, $\gamma \to \tau^{\infty}/G_1$ in material 1 and $\gamma \to \tau^{\infty}/G_2$ in material 2;
(II) continuity of traction on the interface dictates that τ_S and m are continuous;
(III) we assume that no work is done at the interface. By the virtual work relation (7), the work done at an interface of surface S is $\int_S [T_i \Delta u_i + q_i \Delta\theta_i]\,dS$ where T_i is the stress traction and q_i is the couple stress traction on the interface; the jump in displacement is Δu_i and the jump in rotation is $\Delta\theta_i$ along the interface. Thus the supposition that no work is done at the interface implies that both u_1

and θ_3 are continuous at the interface. Since $\theta_3 = -\frac{1}{2}\gamma$ we conclude that γ is continuous at the interface.

The higher order theory requires an extra boundary condition in addition to the standard boundary condition of continuity of displacement at the interface. It is not clear whether continuity of θ_3 is the most appropriate boundary condition to take on physical grounds. This choice has the merit that no work is then done by the couple stress traction at the interface. Other choices of boundary condition are possible, however. In general, the other choices would be associated with a jump in θ_3 at the interface.

The solution to (27a, b) with the boundary conditions (I–III) is

$$\gamma = \frac{\tau^\infty}{G_1}\left(1 - \frac{G_2 l_2}{G_1 l_1 + G_2 l_2}\frac{G_2 - G_1}{G_2}\,\mathrm{e}^{-\sqrt{2}x_2/l_1}\right) \tag{28a}$$

in material 1, and

$$\gamma = \frac{\tau^\infty}{G_2}\left(1 - \frac{G_1 l_1}{G_1 l_1 + G_2 l_2}\frac{G_1 - G_2}{G_1}\,\mathrm{e}^{\sqrt{2}x_2/l_2}\right) \tag{28b}$$

in material 2. Along the interface $x_2 = 0$ and both (28a) and (28b) reduce to $\gamma = \tau^\infty(l_1 + l_2)/(G_1 l_1 + G_2 l_2)$. For the general case of finite G_1, G_2, l_1 and l_2, the boundary layer has an exponential decay length of $l_1/\sqrt{2}$ in material 1 and $l_2/\sqrt{2}$ in material 2. A typical solution is sketched in Fig. 2(a). The shear strain γ in the more compliant layer is reduced near to the interface with the adjacent stiffer layer; similarly the magnitude of γ in the stiffer layer is elevated in the vicinity of the interface with the more compliant layer.

It is instructive to consider the limiting solutions (28a, b). In the limit of $l_1 \to 0$ or $l_2 \to 0$ one of the two solids is unable to carry couple stresses and the shear strain is $\gamma = \tau^\infty/G_1$ throughout solid 1 and $\gamma = \tau^\infty/G_2$ in solid 2. This is the classical elasticity result and shows a jump in γ but not τ_S at the interface.

Now take the limit $G_2 \to \infty$ with G_1, l_1 and l_2 finite. Then, solid 2 can support couple stresses and m is finite at the interface. The shear strain γ vanishes in solid 2 as expected, and in solid 1 γ is given by

$$\gamma = \frac{\tau^\infty}{G_1}\left(1 - \mathrm{e}^{-\sqrt{2}x_2/l_1}\right). \tag{29}$$

Again, the thickness of the boundary layer in solid 1 is given by the decay distance $l_1/\sqrt{2}$. The effective curvature $\chi_e = \sqrt{\frac{2}{3}}|\chi_{32}| = (1/\sqrt{6})|\gamma_{,2}|$ and the strain measure $\mathscr{E} = \sqrt{\varepsilon_e^2 + l_1^2\chi_e^2}$ in solid 1 are

$$\chi_e = \frac{\tau^\infty}{\sqrt{3}G_1 l_1}\,\mathrm{e}^{-\sqrt{2}x_2/l_1} \tag{30}$$

and

$$\mathscr{E} = \frac{\tau^\infty}{\sqrt{3}G_1}\left(1 - 2\,\mathrm{e}^{-\sqrt{2}x_2/l_1} + 2\,\mathrm{e}^{-2\sqrt{2}x_2/l_1}\right)^{1/2} \tag{31}$$

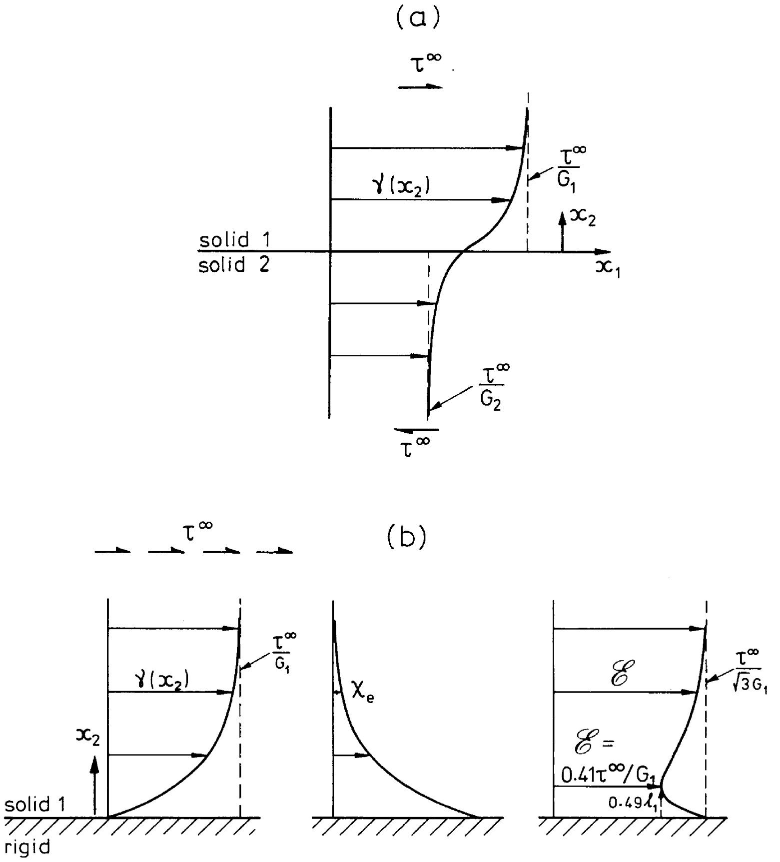

FIG. 2. A bimaterial interface under remote simple shear. (a) Typical form of the boundary layer at the interface between two dissimilar elastic solids. (b) The nature of the boundary layer near the interface between an elastic solid and a rigid solid.

respectively. These distributions $\gamma(x_2)$, $\chi_e(x_2)$ and $\mathscr{E}(x_2)$ are sketched in Fig. 2(b). They have the following physical implications when interpreted for a dislocated solid.

The density ρ_S of statistically stored dislocations scales approximately with $|\gamma|$ and the density ρ_G of geometrically necessary dislocations scales approximately with χ_e. Thus near the rigid boundary ρ_S is lowered and ρ_G is raised. The combined total density of dislocations scales with $\mathscr{E}$; a small reduction in $\mathscr{E}$ exists near to the rigid boundary [with a minimum value of $\tau^\infty/(\sqrt{6}G_1)$ at $x_2 = (\ln 2/\sqrt{2})l_1 = 0.490l_1$)]. We know from dislocation theory that rigid boundaries repel dislocations due to a repulsive image force. Thus we might expect the dislocation density to be lowered near to a boundary. This effect is mimicked in an approximate manner by the predictions of the strain gradient theory.

3. Flow Theory Version

In this section we first review conventional J_2 flow theory for an elastic–plastic solid. A strain gradient version of J_2 flow theory is then proposed. Stability and minimum principles follow in a straightforward fashion. Briefly, the strain gradient version of J_2 flow theory is generated by following the prescription given above for deformation theory. In the presence of couple stresses, the deviatoric, symmetric Cauchy stress $\mathbf{s}$ is replaced by the 13-dimensional vector $\mathbf{\Sigma}$ and the plastic strain rate $\dot{\boldsymbol{\varepsilon}}^{\mathrm{pl}}$ is replaced by the 13-dimensional vector $\dot{\mathscr{E}}^{\mathrm{pl}}$.

In conventional J_2 flow theory, couple stresses are neglected and the strain tensor $\boldsymbol{\varepsilon}$ is decomposed additively into an elastic part $\boldsymbol{\varepsilon}^{\mathrm{el}}$ and a plastic part $\boldsymbol{\varepsilon}^{\mathrm{pl}}$. The elastic strain is related to the Cauchy stress $\boldsymbol{\sigma}$ via the linear relation

$$\varepsilon_{ij}^{\mathrm{el}} = \mathscr{M}_{ijkl}\sigma_{kl}, \tag{32a}$$

where

$$\mathscr{M}_{ijkl} = \frac{(1+\nu)}{2E}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) - \frac{\nu}{E}\delta_{ij}\delta_{kl}. \tag{32b}$$

Here, E is Young's modulus and ν is Poisson's ratio. We note in passing that the inverse of $\mathscr{M}$ is given by $\mathscr{L} = \mathscr{M}^{-1} = \partial^2 w/\partial\boldsymbol{\varepsilon}^2$ where, in the absence of couple stresses, w is defined by the reduced form of (17) with vanishing l.

The plasticity relations of conventional J_2 flow theory provide a connection between the plastic strain rate $\dot{\boldsymbol{\varepsilon}}^{\mathrm{pl}}$ and the stress rate $\dot{\boldsymbol{\sigma}}$; the plastic strain $\boldsymbol{\varepsilon}^{\mathrm{pl}}$ is determined by integration of $\dot{\boldsymbol{\varepsilon}}^{\mathrm{pl}}$ with respect to time. In J_2 theory, $\dot{\boldsymbol{\varepsilon}}^{\mathrm{pl}}$ is taken to be incompressible, and the yield surface Φ is written as

$$\Phi(\boldsymbol{\sigma}, Y) = \sigma_{\mathrm{e}} - Y = 0, \tag{33}$$

where σ_{e} is the von Mises effective stress, $\sigma_{\mathrm{e}} \equiv \sqrt{\frac{3}{2}s_{ij}s_{ij}}$, and Y is the current flow stress. For a hardening solid, the material response is plastic when $\Phi = 0$ and $\dot{\sigma}_{\mathrm{e}} > 0$; and the response is elastic when $\Phi < 0$, or $\Phi = 0$ and $\dot{\sigma}_{\mathrm{e}} \leqslant 0$. The plastic strain rate $\dot{\boldsymbol{\varepsilon}}^{\mathrm{pl}}$ is assumed to be linear in the stress rate $\dot{\boldsymbol{\sigma}}$, and to lie normal to the current yield surface, giving

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{pl}} = \frac{1}{h(\sigma_{\mathrm{e}})}\frac{\partial\Phi}{\partial\boldsymbol{\sigma}}\dot{\sigma}_{\mathrm{e}}, \tag{34}$$

where the hardening rate h is chosen so that the uniaxial tensile response is reproduced. This dictates that h equals the tangent modulus of the stress versus plastic strain curve in simple tension. The work rate $\dot{U}$ per unit volume of the elastic–plastic body is

$$\dot{U} = \sigma_{ij}\dot{\varepsilon}_{ij} = \sigma_{ij}\dot{\varepsilon}_{ij}^{\mathrm{el}} + s_{ij}\dot{\varepsilon}_{ij}^{\mathrm{pl}} \tag{35}$$

and so $\dot{U}$ may be partitioned into an elastic part $\dot{U}^{\mathrm{el}} = \sigma_{ij}\dot{\varepsilon}_{ij}^{\mathrm{el}}$ and a plastic part $\dot{U}^{\mathrm{pl}} = s_{ij}\dot{\varepsilon}_{ij}^{\mathrm{pl}}$. Substitution of (34) into $\dot{U}^{\mathrm{pl}} = s_{ij}\dot{\varepsilon}_{ij}^{\mathrm{pl}}$ gives $\dot{U}^{\mathrm{pl}} = \sigma_{\mathrm{e}}\dot{\sigma}_{\mathrm{e}}/h$ which may be rewritten as $\dot{U}^{\mathrm{pl}} = \sigma_{\mathrm{e}}\dot{\varepsilon}_{\mathrm{e}}^{\mathrm{pl}}$ where the effective plastic strain rate $\dot{\varepsilon}_{\mathrm{e}}^{\mathrm{pl}} \equiv \dot{\sigma}_{\mathrm{e}}/h$. Observe that $\dot{\varepsilon}_{\mathrm{e}}^{\mathrm{pl}} = \sqrt{\frac{2}{3}\dot{\varepsilon}_{ij}^{\mathrm{pl}}\dot{\varepsilon}_{ij}^{\mathrm{pl}}}$ by direct evaluation, making use of (34).

Now assume the existence of couple stresses in the elastic–plastic body. The elastic strain state $(\boldsymbol{\varepsilon}^{\mathrm{el}}, \boldsymbol{\chi}^{\mathrm{el}})$ is assumed to be related to the stress state $(\boldsymbol{\sigma}, \mathbf{m})$ via the elastic strain energy density w^{el}, giving via (15) and (17)

$$\boldsymbol{\sigma} = \mathscr{L} : \boldsymbol{\varepsilon}^{\mathrm{el}} = \frac{\partial w^{\mathrm{el}}}{\partial\boldsymbol{\varepsilon}^{\mathrm{el}}} \tag{36}$$

and

$$l^{-1}\mathbf{m}^{\mathrm{T}} = \mathscr{I} : l\boldsymbol{\chi}^{\mathrm{el}} = \frac{\partial w^{\mathrm{el}}}{l\partial \boldsymbol{\chi}^{\mathrm{el}}}, \tag{37}$$

where

$$w^{\mathrm{el}} = \frac{E}{2(1+\nu)}\left(\frac{\nu}{1-2\nu}(\varepsilon_{kk}^{\mathrm{el}})^2 + \varepsilon_{ij}^{\mathrm{el}}\varepsilon_{ij}^{\mathrm{el}} + l_{\mathrm{el}}^2\chi_{ij}^{\mathrm{el}}\chi_{ij}^{\mathrm{el}}\right). \tag{38}$$

The length scale l_{el} has no physical significance and is introduced in order to partition the curvature tensor $\boldsymbol{\chi}$ into its elastic part $\chi_{im}^{\mathrm{el}} = e_{ijk}\varepsilon_{km,j}^{\mathrm{el}}$ and plastic part $\chi_{im}^{\mathrm{pl}} = e_{ijk}\varepsilon_{km,j}^{\mathrm{pl}}$. A sensible strategy is to take $l_{\mathrm{el}} \ll l$ so that the dominant size effect is associated with plastic rather than elastic strain gradients. Explicit expressions for the elastic moduli $\mathscr{L}$ and $\mathscr{I}$ are obtained by differentiation of (38) with respect to $\boldsymbol{\varepsilon}^{\mathrm{el}}$ and $l\boldsymbol{\chi}^{\mathrm{el}}$, respectively, giving

$$\mathscr{L}_{ijkl} = \frac{E}{2(1+\nu)}\left(\frac{2\nu}{1-2\nu}\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \tag{39}$$

and

$$\mathscr{I}_{ijkl} = \frac{E}{(1+\nu)}\left(\frac{l_{\mathrm{el}}}{l}\right)^2\delta_{ik}\delta_{jl}. \tag{40}$$

The elastic strain state is obtained from the stress state by inversion of (37)–(40)

$$\boldsymbol{\varepsilon}^{\mathrm{el}} = \mathscr{M} : \boldsymbol{\sigma}, \tag{41}$$

where $\mathscr{M} = \mathscr{L}^{-1}$ and in component form is

$$\mathscr{M}_{ijkl} = \frac{(1+\nu)}{2E}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{\nu}{E}\delta_{ij}\delta_{kl}. \tag{42}$$

Similarly,

$$l\boldsymbol{\chi}^{\mathrm{el}} = \mathscr{K} : l^{-1}\mathbf{m}^{\mathrm{T}}, \tag{43}$$

where $\mathscr{K}$ is the inverse of $\mathscr{I}$ and has the components

$$\mathscr{K}_{ijkl} = \frac{(1+\nu)}{E}\left(\frac{l}{l_{\mathrm{el}}}\right)^2\delta_{ik}\delta_{jl}. \tag{44}$$

A prescription is now given for the dependence of the plastic strain rate upon the stress rate in the presence of couple stresses. The argument parallels that given for the deformation theory solid in the discussion following (13). In the presence of couple stresses, the deviatoric, symmetric Cauchy stress $\mathbf{s}$ is replaced by the 13-dimensional vector $\boldsymbol{\Sigma} = (\mathbf{s}, l^{-1}\mathbf{m}^{\mathrm{T}})$ comprising the five components of $\mathbf{s}$ and the eight components of $l^{-1}\mathbf{m}^{\mathrm{T}}$. Similarly, the plastic strain rate $\dot{\boldsymbol{\varepsilon}}^{\mathrm{pl}}$ is replaced by the 13-dimensional vector $\dot{\mathscr{E}}^{\mathrm{pl}} = (\dot{\varepsilon}^{\mathrm{pl}}, l\dot{\chi})$. The yield surface (33) generalizes to

$$\Phi(\boldsymbol{\Sigma}, Y) = \Sigma - Y = 0, \tag{45}$$

where the overall effective stress Σ is defined by

$$\Sigma^2 = \tfrac{3}{2}\mathbf{\Sigma}\cdot\mathbf{\Sigma} = \tfrac{3}{2}s_{ij}s_{ij} + \tfrac{3}{2}l^{-2}m_{ij}m_{ij} = \sigma_e^2 + l^{-2}m_e^2, \tag{46}$$

in agreement with (12). When the solid is subjected to a uniaxial tensile stress $\sigma, \Sigma = \sigma$ and yield occurs when $\sigma = Y$ by (45); we interpret Y as the uniaxial flow stress. Equation (45) is a natural generalization of (33) once it is assumed that $\mathbf{s}$ is replaced by $\mathbf{\Sigma} = (\mathbf{s}, l^{-1}\mathbf{m}^T)$ and σ_e by Σ in the couple stress version of the theory.

Plastic straining is assumed to be normal to the yield surface and the plastic strain rate is taken to be linear in the stress rate: (34) then generalizes to

$$\dot{\mathcal{E}}^{pl} = \frac{1}{h(\Sigma)}\frac{\partial \Phi}{\partial \mathbf{\Sigma}}\dot{\Sigma}, \tag{47}$$

where $\dot{\mathcal{E}}^{pl} = (\dot{\varepsilon}^{pl}, l\dot{\chi})$ has already been defined. In the case of uniaxial tension, where the axial stress is σ and the plastic strain is ε^{pl}, we find $\Sigma = \sigma$ and (47) reduces to

$$\dot{\varepsilon}^{pl} = \frac{\dot{\sigma}}{h(\sigma)}. \tag{48}$$

Thus the interpretation of h remains the same as for conventional J_2 flow theory.

The plastic work rate $\dot{U}^{pl}$ is, via (10),

$$\dot{U}^{pl} = s_{ij}\dot{\varepsilon}_{ij}^{pl} + m_{ji}\dot{\chi}_{ij}^{pl} = \mathbf{\Sigma}\cdot\dot{\mathcal{E}}^{pl}. \tag{49}$$

On substitution of the expression (47) for $\dot{\mathcal{E}}^{pl}$ into (49) we get $\dot{U}^{pl} = \Sigma\dot{\Sigma}/h$ which may be rewritten as $\dot{U}^{pl} = \Sigma\dot{\mathcal{E}}^{pl}$ where the overall effective plastic strain rate $\dot{\mathcal{E}}^{pl} \equiv \dot{\Sigma}/h$. An alternative expression for $\dot{\mathcal{E}}^{pl}$ is obtained by evaluation of the invariant $\sqrt{\tfrac{2}{3}\dot{\mathcal{E}}^{pl}\cdot\dot{\mathcal{E}}^{pl}}$ using (47), to give

$$\dot{\mathcal{E}}^{pl} = \frac{\dot{\Sigma}}{h} = \sqrt{\tfrac{2}{3}\dot{\mathcal{E}}^{pl}\cdot\dot{\mathcal{E}}^{pl}}. \tag{50}$$

For completeness, we introduce the effective plastic curvature rate $\dot{\chi}_e^{pl}$ as $\dot{\chi}_e^{pl} = \sqrt{\tfrac{2}{3}\dot{\chi}_{ij}^{pl}\dot{\chi}_{ij}^{pl}}$ and note that $\dot{\mathcal{E}}^{pl}$ may be written in terms of $\dot{\varepsilon}_e^{pl}$ and $\dot{\chi}_e^{pl}$ as

$$\dot{\mathcal{E}}^{pl} = \sqrt{\tfrac{2}{3}\dot{\mathcal{E}}^{pl}\cdot\dot{\mathcal{E}}^{pl}} = \sqrt{\tfrac{2}{3}\dot{\varepsilon}_{ij}^{pl}\dot{\varepsilon}_{ij}^{pl} + \tfrac{2}{3}l^2\dot{\chi}_{ij}^{pl}\dot{\chi}_{ij}^{pl}} = \sqrt{(\dot{\varepsilon}_e^{pl})^2 + (l\dot{\chi}_e^{pl})^2}. \tag{51}$$

3.1. *Summary of elastic–plastic constitutive relations*

The main constitutive relations in the strain gradient formulation are now summarized in index notation. Plastic flow is normal to the yield surface such that

$$\dot{\varepsilon}_{ij}^{pl} = \frac{3}{2h}\frac{s_{ij}}{\Sigma}\dot{\Sigma} \tag{52a}$$

and

$$l\dot{\chi}_{ij}^{pl} = \frac{3}{2h}\frac{l^{-1}m_{ji}}{\Sigma}\dot{\Sigma}, \tag{52b}$$

by (47). The rate of overall effective stress $\dot{\Sigma}$ is given by the rate form of (46),

$$\dot{\Sigma} = \frac{3}{2}\frac{s_{ij}}{\Sigma}\dot{s}_{ij} + \frac{3}{2}\frac{l^{-1}m_{ji}}{\Sigma}l^{-1}\dot{m}_{ji}. \tag{52c}$$

The elastic strain rates follow from (41) and (43) as

$$\dot{\varepsilon}_{ij}^{\mathrm{el}} = \mathscr{M}_{ijkl}\dot{\sigma}_{kl} \tag{53}$$

and

$$l\dot{\chi}_{ij}^{\mathrm{el}} = \mathscr{K}_{ijkl}l^{-1}\dot{m}_{lk}, \tag{54}$$

with $\mathscr{M}_{ijkl}$ and $\mathscr{K}_{ijkl}$ given by (42) and (44) respectively. Note that the current formulation predicts that couple stresses remain present in the case of a purely elastic response with vanishing plastic strains. This is for purely mathematical convenience, and is given no physical significance. Indeed, the magnitude of the elastic couple stresses may be made arbitrarily small by choosing the ratio l_{el}/l to be sufficiently small.

In the above strain gradient versions of J_2 deformation theory and J_2 flow theory, proportional loading occurs at a material point when all stress components of $(\boldsymbol{\sigma}, \mathbf{m})$ lie along a fixed direction which is written as $(\boldsymbol{\sigma}^0, \mathbf{m}^0)$; the components of $(\boldsymbol{\sigma}, \mathbf{m})$ then scale in magnitude with a monotonically increasing scalar quantity, λ, such that $(\boldsymbol{\sigma}, \mathbf{m}) = \lambda(\boldsymbol{\sigma}^0, \mathbf{m}^0)$. When proportional loading is experienced by a material point the predictions of the above strain gradient versions of J_2 deformation theory and J_2 flow theory coincide.

3.2. *Minimum principles*

The yield surface (45) is convex in the stress space $(\boldsymbol{\sigma}, \mathbf{m})$ and the plastic strain rate is normal to the yield surface. Hence the strain gradient version of J_2 flow theory (with $h > 0$) satisfies the slightly more generalized form of Drucker's stability postulates (DRUCKER, 1951)

$$\dot{\sigma}_{ij}\dot{\varepsilon}_{ij}^{\mathrm{pl}} + \dot{m}_{ji}\dot{\chi}_{ij}^{\mathrm{pl}} \geqslant 0 \tag{55a}$$

for a stress rate $(\dot{\boldsymbol{\sigma}}, \dot{\mathbf{m}})$ corresponding to a plastic strain rate $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{pl}}, \dot{\boldsymbol{\chi}}^{\mathrm{pl}})$, and

$$(\sigma_{ij} - \sigma_{ij}^*)\dot{\varepsilon}_{ij}^{\mathrm{pl}} + (m_{ji} - m_{ji}^*)\dot{\chi}_{ij}^{\mathrm{pl}} \geqslant 0 \tag{55b}$$

for a stress state $(\boldsymbol{\sigma}, \mathbf{m})$ associated with a plastic strain rate $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{pl}}, \dot{\boldsymbol{\chi}}^{\mathrm{pl}})$, and a neighbouring stress state $(\boldsymbol{\sigma}^*, \mathbf{m}^*)$ on or within the yield surface.

Minimum principles are now given for the displacement rate and for the stress rate, for the strain gradient version of J_2 flow theory. These minimum principles follow directly from those outlined by KOITER (1960) for phenomenological plasticity theories with multiple yield functions, and from the minimum principles given in more general form by HILL (1966) for a metal crystal deforming in multislip. The presence of couple stresses can be included simply by replacing $\mathbf{s}$ by $\boldsymbol{\Sigma}$ and $\dot{\boldsymbol{\varepsilon}}^{\mathrm{pl}}$ by $\dot{\boldsymbol{\mathscr{E}}}^{\mathrm{pl}}$, as outlined above.

Consider a body of volume V and surface S comprised of an elastic–plastic solid which obeys the strain gradient version of J_2 flow theory (52)–(54). The body is loaded by the instantaneous stress traction rate $\dot{T}_i^0$ and couple stress traction rate $\dot{q}_i^0$ on a portion S_{T} of the surface. The velocity is prescribed as $\dot{u}_i^0$ and the rotation rate is $\dot{\theta}_i^0$ on the remaining portion S_u of the surface. Then the following minimum principles may be stated.

Minimum principle for the displacement rate. Consider all admissible velocity fields $\dot{u}_i$ which satisfy $\dot{u}_i = \dot{u}_i^0$ and $\dot{\theta}_i = \frac{1}{2}e_{ijk}\dot{u}_{k,j} = \dot{\theta}_i^0$ on S_u. Let $\dot{\varepsilon}_{ij} \equiv \frac{1}{2}(\dot{u}_{i,j}+\dot{u}_{j,i})$ and $\dot{\chi}_{ip} \equiv \frac{1}{2}e_{ijk}\dot{u}_{k,jp}$ be the state of strain rate derived from $\dot{u}_i$, and define $(\dot{\boldsymbol{\sigma}}, \dot{\mathbf{m}})$ to be the stress rate field associated with $(\dot{\boldsymbol{\varepsilon}}, \dot{\boldsymbol{\chi}})$ via the constitutive law for the strain gradient version of J_2 flow theory (52)–(54). Then, the functional $F(\dot{\mathbf{u}})$, defined by

$$F(\dot{\mathbf{u}}) = \frac{1}{2}\int_V [\dot{\sigma}_{ij}\dot{\varepsilon}_{ij} + \dot{m}_{ji}\dot{\chi}_{ij}]\,\mathrm{d}V - \int_{S_\mathrm{T}} [\dot{T}_i^0\dot{u}_i + \dot{q}_i^0\dot{\theta}_i]\,\mathrm{d}S \tag{56}$$

is minimized by the exact solution $(\dot{\mathbf{u}}, \dot{\boldsymbol{\varepsilon}}, \dot{\boldsymbol{\chi}}, \dot{\boldsymbol{\sigma}}, \dot{\mathbf{m}})$. The exact solution is unique since the minimum is absolute.

Minimum principle for the stress rate. Consider instead all admissible equilibrium stress rate fields $(\dot{\boldsymbol{\sigma}}, \dot{\mathbf{m}})$ which satisfy the traction boundary conditions $(\dot{\sigma}_{ij}+\dot{\tau}_{ij})n_i = \dot{T}_j^0$ and $\dot{m}_{ij}n_i = \dot{q}_j^0$ on S_T. Let $\dot{u}_i^0$ and $\dot{\theta}_i^0$ be prescribed on the remaining portion S_u, and define $(\dot{\boldsymbol{\varepsilon}}, \dot{\boldsymbol{\chi}})$ to be the state of strain rate associated with the stress rate $(\dot{\boldsymbol{\sigma}}, \dot{\mathbf{m}})$ via the constitutive law (52)–(54). Then, the functional $H(\dot{\boldsymbol{\sigma}}, \dot{\mathbf{m}})$, defined by,

$$H(\dot{\boldsymbol{\sigma}}, \dot{\mathbf{m}}) = \frac{1}{2}\int_V [\dot{\sigma}_{ij}\dot{\varepsilon}_{ij} + \dot{m}_{ji}\dot{\chi}_{ij}]\,\mathrm{d}V - \int_{S_u} [(\dot{\sigma}_{ij}+\dot{\tau}_{ij})n_i\dot{u}_j^0 + \dot{m}_{ij}n_i\dot{\theta}_j^0]\,\mathrm{d}S \tag{57}$$

is minimized by the exact solution $(\dot{\mathbf{u}}, \dot{\boldsymbol{\varepsilon}}, \dot{\boldsymbol{\chi}}, \dot{\boldsymbol{\sigma}}, \dot{\mathbf{m}})$. Uniqueness follows directly from the statement that the minimum is absolute.

The proofs of the minimum principles for the displacement rate and stress rate require three fundamental inequalities, which are the direct extensions of those given by KOITER (1960) and HILL (1966), and are stated here without proof. Assume that at each material point a stress state $(\boldsymbol{\sigma}, \mathbf{m})$ is known; the material may, or may not, be at yield. Let $(\dot{\boldsymbol{\varepsilon}}, \dot{\boldsymbol{\chi}})$ be associated with any assumed $(\dot{\boldsymbol{\sigma}}, \dot{\mathbf{m}})$ via the constitutive law (52)–(54). Similarly, let $(\dot{\boldsymbol{\varepsilon}}^*, \dot{\boldsymbol{\chi}}^*)$ be associated with an alternative stress rate field $(\dot{\boldsymbol{\sigma}}^*, \dot{\mathbf{m}}^*)$. Then, the three inequalities are

$$(\dot{\varepsilon}_{ij} - \dot{\varepsilon}_{ij}^*)(\dot{\sigma}_{ij} - \dot{\sigma}_{ij}^*) + (\dot{\chi}_{ij} - \dot{\chi}_{ij}^*)(\dot{m}_{ji} - \dot{m}_{ji}^*) \geqslant 0, \tag{58a}$$

$$(\dot{\varepsilon}_{ij}^*\dot{\sigma}_{ij}^* + \dot{\varepsilon}_{ij}\dot{\sigma}_{ij} - 2\dot{\varepsilon}_{ij}^*\dot{\sigma}_{ij}) + (\dot{\chi}_{ij}^*\dot{m}_{ji}^* + \dot{\chi}_{ij}\dot{m}_{ji} - 2\dot{\chi}_{ij}^*\dot{m}_{ji}) \geqslant 0 \tag{58b}$$

and

$$(\dot{\varepsilon}_{ij}^*\dot{\sigma}_{ij}^* + \dot{\varepsilon}_{ij}\dot{\sigma}_{ij} - 2\dot{\varepsilon}_{ij}\dot{\sigma}_{ij}^*) + (\dot{\chi}_{ij}^*\dot{m}_{ji}^* + \dot{\chi}_{ij}\dot{m}_{ji} - 2\dot{\chi}_{ij}\dot{m}_{ji}^*) \geqslant 0. \tag{58c}$$

The equality sign holds in the above three expressions if and only if $\dot{\boldsymbol{\sigma}}^* = \dot{\boldsymbol{\sigma}}$ and $\dot{\mathbf{m}}^* = \dot{\mathbf{m}}$.

4. CONSTITUTIVE POTENTIAL FOR A DILUTE CONCENTRATION OF INCLUSIONS

HILL (1967) and RICE (1970) have developed techniques for estimating the macroscopic average response of a heterogeneous material, based on the response at each material point. In the same spirit, DUVA and HUTCHINSON (1984) derived constitutive relations for a power law creeping body containing a dilute concentration of voids, and DUVA (1984) estimated the stiffening of a power law material due to the presence of a dilute (and non-dilute) concentration of rigid spherical particles. The basic approach is to define in a rigorous fashion a constitutive potential for the body containing a dilute concentration of inclusions in terms of the change in potential due to the introduction of an isolated inclusion in the matrix material. The development given below is a generalization of that given by DUVA and HUTCHINSON (1984) for the case of a solid which can support couple stresses.

The formulation is done within the context of deformation theory. A power law stress–strain relation is taken for the matrix, and for the kernel problem of an isolated inclusion in an infinite matrix remote proportional loading is applied. These stipulations allow for a generalization of Illuyshin's theorem to be enforced: proportional loading occurs at each material point within the body and results for deformation theory coincide exactly with the predictions of flow theory.

We consider as a macroscopic representative volume element a block of material with volume V consisting of a dilute concentration ρ of inclusions in an incompressible nonlinear matrix. Specifically, the matrix is taken to be a power law deformation theory plastic solid with constitutive description (8)–(13), and the inclusions are either traction-free voids or bonded rigid particles. In the sequel, we shall use the term "inclusion" to refer to either voids or rigid particles. The matrix material is characterized by a potential of the stress $\phi(\boldsymbol{\sigma}, \mathbf{m})$, where

$$\phi(\boldsymbol{\sigma}, \mathbf{m}) = \int_0^{\boldsymbol{\Sigma}} \mathscr{E} \cdot \mathrm{d}\boldsymbol{\Sigma} = \int_0^{\mathbf{s}} \varepsilon_{ij}\, \mathrm{d}s_{ij} + \int_0^{l^{-1}\mathbf{m}^{\mathrm{T}}} l\chi_{ij}\, \mathrm{d}(l^{-1}m_{ji}) = \frac{\Sigma_0 \mathscr{E}_0}{(n+1)} \left(\frac{\Sigma}{\Sigma_0}\right)^{n+1}, \tag{59}$$

so that the strain at a material point in the matrix is

$$\mathscr{E} = \frac{\partial \phi}{\partial \boldsymbol{\Sigma}}; \quad \varepsilon_{ij} = \frac{\partial \phi}{\partial s_{ij}}; \quad l\chi_{ij} = \frac{\partial \phi}{l^{-1}\partial m_{ji}}. \tag{60}$$

The dual potential of $\phi(\boldsymbol{\sigma}, \mathbf{m})$ is the strain energy density function $w(\boldsymbol{\varepsilon}, \boldsymbol{\chi})$, defined by

$$w(\boldsymbol{\varepsilon}, \boldsymbol{\chi}) = \int_0^{\mathscr{E}} \boldsymbol{\Sigma} \cdot \mathrm{d}\mathscr{E} = \int_0^{\boldsymbol{\varepsilon}} s_{ij}\, \mathrm{d}\varepsilon_{ij} + \int_0^{l\boldsymbol{\chi}} l^{-1} m_{ji}\, \mathrm{d}(l\chi_{ij}) = \frac{n}{(n+1)} \Sigma_0 \mathscr{E}_0 \left(\frac{\mathscr{E}}{\mathscr{E}_0}\right)^{(n+1)/n}, \tag{61}$$

so that

$$\boldsymbol{\Sigma} = \frac{\partial w}{\partial \mathscr{E}}; \quad s_{ij} = \frac{\partial w}{\partial \varepsilon_{ij}}; \quad l^{-1} m_{ji} = \frac{\partial w}{l \partial \chi_{ij}}. \tag{62}$$

Note that

$$\phi(\boldsymbol{\sigma}, \mathbf{m}) + w(\boldsymbol{\varepsilon}, \boldsymbol{\chi}) = \boldsymbol{\Sigma} \cdot \mathscr{E} = s_{ij}\varepsilon_{ij} + m_{ji}\chi_{ij} = \Sigma \mathscr{E} = \sigma_e \varepsilon_e + m_e \chi_e. \tag{63}$$

4.1. *The macroscopic potential*

Let $(\bar{\boldsymbol{\sigma}}, \bar{\mathbf{m}})$ and $(\bar{\boldsymbol{\varepsilon}}, \bar{\boldsymbol{\chi}})$ denote the macroscopic, or average, stress and strain state of a representative block of material of volume V containing a distribution of either voids or rigid particles. The macroscopic constitutive potential $\Phi(\bar{\boldsymbol{\sigma}}, \bar{\mathbf{m}})$ of the block provides the macroscopic strain according to

$$\bar{\varepsilon}_{ij} = \frac{\partial \Phi}{\partial \bar{\sigma}_{ij}}; \quad l\bar{\chi}_{ij} = \frac{\partial \Phi}{\partial (l^{-1}\bar{m}_{ji})}, \tag{64}$$

where Φ is related to the distribution of local potential by

$$V\Phi(\bar{\boldsymbol{\sigma}}, \bar{\mathbf{m}}) = \int_{V_m} \phi(\boldsymbol{\sigma}, \mathbf{m}) \, dV, \tag{65}$$

with V_m denoting the region occupied by matrix material. There is no contribution to Φ from the voids or rigid particles since the pointwise potential ϕ vanishes for both voids and rigid particles.

For a dilute concentration of inclusions with volume fraction ρ the macroscopic potential can be written as

$$\Phi(\bar{\boldsymbol{\sigma}}, \bar{\mathbf{m}}) = \phi(\bar{\boldsymbol{\sigma}}, \bar{\mathbf{m}}) + \rho \Phi_I(\bar{\boldsymbol{\sigma}}, \bar{\mathbf{m}}), \tag{66}$$

where $\Phi_I(\bar{\boldsymbol{\sigma}}, \bar{\mathbf{m}})$ is the change in potential due to the introduction of an inclusion of *unit* volume into an infinite block of matrix material that is subjected to the remote uniform stress $\boldsymbol{\sigma}^\infty = \bar{\boldsymbol{\sigma}}$, $\mathbf{m}^\infty = \bar{\mathbf{m}}$.

To define $\Phi_I(\bar{\boldsymbol{\sigma}}, \bar{\mathbf{m}})$ consider an isolated inclusion of volume V_I centred in a spherical matrix of volume V_m and of finite outer radius R, as shown in Fig. 3. Uniform stress tractions $T_j^\infty = (\sigma_{ij}^\infty + \tau_{ij}^\infty) n_i$ and couple stress tractions $q_j^\infty = m_{ij}^\infty n_i$ are applied to the outer surface of the matrix. [Since $\mathbf{m}^\infty$ is taken to be uniform $\bar{\boldsymbol{\tau}} = \boldsymbol{\tau}^\infty$ vanishes by (4).] Define the change in potential $V_I \Phi_I(\boldsymbol{\sigma}^\infty, \mathbf{m}^\infty)$ due to the introduction of the inclusion by

$$V_I \Phi_I(\boldsymbol{\sigma}^\infty, \mathbf{m}^\infty) = \int_{V_m} [\phi(\boldsymbol{\sigma}, \mathbf{m}) - \phi(\boldsymbol{\sigma}^\infty, \mathbf{m}^\infty)] \, dV - V_I \phi(\boldsymbol{\sigma}^\infty, \mathbf{m}^\infty). \tag{67}$$

This expression may be rearranged to a more convenient form using the principle of virtual work and (63) to give

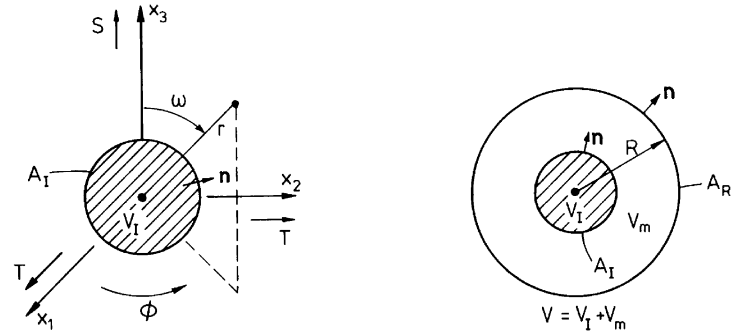

FIG. 3. Geometry and conventions for the isolated inclusion calculations.

$$V_1\Phi_1(\boldsymbol{\sigma}^\infty,\mathbf{m}^\infty) = \int_{V_m} [\phi(\boldsymbol{\sigma},\mathbf{m}) - \phi(\boldsymbol{\sigma}^\infty,\mathbf{m}^\infty) - \tilde{\sigma}_{ij}\varepsilon_{ij}^\infty - \tilde{m}_{ji}\chi_{ij}^\infty]\,\mathrm{d}V + \int_{A_1} [\tilde{T}_j u_j^\infty + \tilde{q}_j\theta_j^\infty]\,\mathrm{d}S - V_1\phi(\boldsymbol{\sigma}^\infty,\mathbf{m}^\infty), \quad (68)$$

where the tilde symbol denotes the change in quantity due to introduction of the inclusion,

$$\tilde{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma} - \boldsymbol{\sigma}^\infty; \quad \tilde{\mathbf{m}} \equiv \mathbf{m} - \mathbf{m}^\infty; \quad \tilde{\boldsymbol{\tau}} \equiv \boldsymbol{\tau} - \boldsymbol{\tau}^\infty; \quad \tilde{T}_j \equiv -(\tilde{\sigma}_{ij} + \tilde{\tau}_{ij})n_i \quad \text{and} \quad \tilde{q}_j \equiv -\tilde{m}_{ij}n_i. \quad (69)$$

A negative sign is present in the definitions (69) of $\tilde{T}_j$ and $\tilde{q}_j$ as the unit normal on the surface A_1 of the inclusion is taken to point into the matrix, as illustrated in Fig. 3. The integrand of the volume integral in (68) decays sufficiently rapidly at large distances from the inclusion that Φ_1 can be evaluated either as the limit of the finite problem as the outer radius R becomes unbounded, or directly from the infinite problem where the remote stress is $(\boldsymbol{\sigma}^\infty, \mathbf{m}^\infty)$.

4.2. *Relation between the macroscopic potential and the minimum principle for the boundary value problem*

A minimum principle for the displacements may be used to find approximate Rayleigh–Ritz solutions for an isolated inclusion in an infinite matrix under remote stresses $(\boldsymbol{\sigma}^\infty, \mathbf{m}^\infty)$. The principle for the non-linear solid without couple stresses is from Hill (1956) and was modified by Budiansky *et al.* (1982) to be applicable to infinite regions. Here, the minimum principle is developed for the infinite non-linear body which supports couple stresses. The starting point is (19) for the finite non-linear body containing couple stresses.

Consider a finite block of volume V containing an isolated inclusion of volume V_1 centred in a spherical matrix of volume V_m and of finite outer radius R. The outer surface of the block is subjected to the uniform tractions $T_j^\infty = (\sigma_{ij}^\infty + \tau_{ij}^\infty)n_i$ and $q_j^\infty = m_{ij}^\infty n_i$, where $\mathbf{n}$ is the outward normal to the spherical block. Take $\mathbf{u}$ as any admissible displacement field and $(\boldsymbol{\varepsilon}, \boldsymbol{\chi})$ as the associated strain quantities. Then the actual field in the block minimizes

$$P(\mathbf{u}) = \int_{V_m} w(\boldsymbol{\varepsilon},\boldsymbol{\chi})\,\mathrm{d}V - \int_{A_R} [T_j^\infty u_j + q_j^\infty \theta_j]\,\mathrm{d}S, \quad (70)$$

which is a restatement of (19) in the current notation. Let the displacement field and strain field be decomposed into the uniform remote field (denoted by a superscript ∞) associated with $(\boldsymbol{\sigma}^\infty, \mathbf{m}^\infty)$ and an additional field (denoted by the tilde symbol ~)

$$u_i = \varepsilon_{ij}^\infty x_j + \tilde{u}_i; \quad \theta_i = \chi_{ij}^\infty x_j + \tilde{\theta}_i; \quad \varepsilon_{ij} = \varepsilon_{ij}^\infty + \tilde{\varepsilon}_{ij} \quad \text{and} \quad \chi_{ij} = \chi_{ij}^\infty + \tilde{\chi}_{ij}. \quad (71)$$

The additional field is associated with the presence of the inclusion and vanishes at infinity. The *change* in potential energy upon introducing the inclusion (void or rigid particle) into the spherical block is

$$P_{\mathrm{I}}(\mathbf{u}) = \int_{V_{\mathrm{m}}} [w(\boldsymbol{\varepsilon},\boldsymbol{\chi}) - w(\boldsymbol{\varepsilon}^{\infty},\boldsymbol{\chi}^{\infty})]\,\mathrm{d}V - \int_{A_{\mathrm{R}}} [T_j^{\infty}\tilde{u}_j + q_j^{\infty}\tilde{\theta}_j]\,\mathrm{d}S - V_{\mathrm{I}} w(\boldsymbol{\varepsilon}^{\infty},\boldsymbol{\chi}^{\infty}). \quad (72)$$

Next, note that

$$\int_{A_{\mathrm{R}}} [T_j^{\infty}\tilde{u}_j + q_j^{\infty}\tilde{\theta}_j]\,\mathrm{d}S = -\int_{A_{\mathrm{I}}} [T_j^{\infty}\tilde{u}_j + q_j^{\infty}\tilde{\theta}_j]\,\mathrm{d}S + \int_{V_{\mathrm{m}}} [\sigma_{ij}^{\infty}\tilde{\varepsilon}_{ij} + m_{ji}^{\infty}\tilde{\chi}_{ij}]\,\mathrm{d}V, \quad (73)$$

where $T_j^{\infty} = -(\sigma_{ij}^{\infty} + \tau_{ij}^{\infty})n_i$ and $q_j^{\infty} = -m_{ij}^{\infty}n_i$ on the surface A_{I} of the inclusion since the unit normal $\mathbf{n}$ is taken to point into the matrix there. Substitution of (73) into (72) enables $P_{\mathrm{I}}(\mathbf{u})$ to be written in the more convenient form

$$P_{\mathrm{I}}(\mathbf{u}) = \int_{V_{\mathrm{m}}} [w(\boldsymbol{\varepsilon},\boldsymbol{\chi}) - w(\boldsymbol{\varepsilon}^{\infty},\boldsymbol{\chi}^{\infty}) - \sigma_{ij}^{\infty}\tilde{\varepsilon}_{ij} - m_{ji}^{\infty}\tilde{\chi}_{ij}]\,\mathrm{d}V + \int_{A_{\mathrm{I}}} [T_j^{\infty}\tilde{u}_j + q_j^{\infty}\tilde{\theta}_j]\,\mathrm{d}S - V_{\mathrm{I}} w(\boldsymbol{\varepsilon}^{\infty},\boldsymbol{\chi}^{\infty}). \quad (74)$$

In the limit $R \to \infty$, the above expression for $P_{\mathrm{I}}(\mathbf{u})$ gives the change in potential energy due to the introduction of an inclusion into the infinite block of matrix material. Further, if $P_{\mathrm{I}}(\mathbf{u})$ is regarded as a functional in $\mathbf{u}$ it provides the minimum principle for the infinite region.

The expression (74) for $P_{\mathrm{I}}(\mathbf{u})$ is of the same form as the relation (68) for $V_{\mathrm{I}}\Phi_{\mathrm{I}}(\boldsymbol{\sigma}^{\infty},\mathbf{m}^{\infty})$. In fact, a straightforward connection exists between $V_{\mathrm{I}}\Phi_{\mathrm{I}}(\boldsymbol{\sigma}^{\infty},\mathbf{m}^{\infty})$ and the minimum value $P_{\min}$ of $P_{\mathrm{I}}(\mathbf{u})$. Upon summing (68) and (74) we find, after some manipulation,

$$P_{\min} + V_{\mathrm{I}}\Phi_{\mathrm{I}} = \int_{A_{\mathrm{I}}} [T_j u_j + q_j \theta_j]\,\mathrm{d}S. \quad (75)$$

For a traction-free void $T_j = q_j = 0$ on the void surface, and for a bonded rigid particle we take $u_j = \theta_j = 0$ on the surface of the particle [as discussed more fully in the paragraph following (27)]. Thus, for both types of inclusion, the right-hand side of (75) vanishes, and

$$P_{\min} + V_{\mathrm{I}}\Phi_{\mathrm{I}} = 0. \quad (76)$$

This relation closely parallels (23) and provides a simple way of calculating $V_{\mathrm{I}}\Phi_{\mathrm{I}}(\boldsymbol{\sigma}^{\infty},\mathbf{m}^{\infty})$ directly from the solution to the minimum problem.

5. Strengthening Due to a Dilute Concentration of Rigid Spherical Particles

Consider a macroscopic volume element consisting of a dilute concentration ρ of rigid particles embedded in a power law deformation theory matrix (13). The particles are spherical, and are equi-sized of radius a. The average stress on the macroscopic volume element is taken to be $\bar{\boldsymbol{\sigma}}$ and the average couple stress $\bar{\mathbf{m}}$ is assumed to vanish. This is reasonable provided the macroscopic strain gradients are small over length scales much greater than the microstructural length scale l. The average strain is $\bar{\boldsymbol{\varepsilon}}$ and the average curvature $\bar{\boldsymbol{\chi}}$ vanishes. The local couple stress and curvature in the vicinity of each particle does not vanish, however, and gives rise to a particle size dependence of macroscopic strength. The macroscopic response is given by (64) and (65) in terms of the change in macroscopic potential $\Phi_{\rm I}(\bar{\boldsymbol{\sigma}}, \bar{\mathbf{m}} = \mathbf{0})$. Numerical estimates are given for $\Phi_{\rm I}(\bar{\boldsymbol{\sigma}}, \mathbf{0})$ in Section 5.1 below. First, some deductions are made for the functional form of $\Phi_{\rm I}(\bar{\boldsymbol{\sigma}}, \mathbf{0})$ and the inverted form of the constitutive law $\bar{\boldsymbol{\sigma}} = \bar{\boldsymbol{\sigma}}(\bar{\boldsymbol{\varepsilon}})$.

By isotropy, $\Phi_{\rm I}(\bar{\boldsymbol{\sigma}}, \mathbf{0})$ depends at most upon the three invariants of $\bar{\boldsymbol{\sigma}}$: the mean stress $\bar{\sigma}_{\rm m} \equiv \frac{1}{3}\bar{\sigma}_{kk}$, the von Mises effective stress $\bar{\sigma}_{\rm e} \equiv \sqrt{\frac{3}{2}\bar{s}_{ij}\bar{s}_{ij}}$ (where $\bar{\mathbf{s}}$ is the deviatoric, symmetric part of the macroscopic Cauchy stress), and the third invariant $J_3 = (\bar{s}_{ij}\bar{s}_{jk}\bar{s}_{ki})^{1/3}$. Since the matrix and particle are assumed to be incompressible there is no dependence of $\Phi_{\rm I}$ upon $\bar{\sigma}_{\rm m}$. We will ignore the dependence of $\Phi_{\rm I}$ upon the third invariant as DUVA (1984) has shown it has a relatively minor effect in the limit $l = 0$. Then, because $\Phi_{\rm I}$ is homogeneous of degree $n+1$ in $\boldsymbol{\sigma}$, one can write

$$\Phi_{\rm I} = -\Sigma_0 \mathscr{E}_0 \left(\frac{\bar{\sigma}_{\rm e}}{\Sigma_0}\right)^{n+1} f_{\rm p}(l/a, n), \tag{77}$$

where the kernel problem of an isolated rigid particle is required in order to determine values for the non-dimensional function $f_{\rm p}$. The function $f_{\rm p}$ provides a convenient measure of the strengthening effect due to the presence of rigid particles, and depends upon both the ratio l/a and the strain hardening index n. The change in potential $\Phi_{\rm I}$ is negative for the case of a bonded, rigid particle and so a minus sign has been introduced in (77) to make the function $f_{\rm p}$ positive, for the sake of convenience.

The macroscopic strain $\bar{\boldsymbol{\varepsilon}}$ is given in terms of the macroscopic stress $\bar{\boldsymbol{\sigma}}$ via (64), (66) and (77) as

$$\bar{\varepsilon}_{ij} = \tfrac{3}{2}\mathscr{E}_0 \left(\frac{\bar{\sigma}_{\rm e}}{\Sigma_0}\right)^{n-1} \frac{\bar{s}_{ij}}{\Sigma_0} \{1 - \rho(n+1) f_{\rm p}\}, \tag{78a}$$

which may be inverted directly to give, to leading order in ρ,

$$\bar{s}_{ij} = \tfrac{2}{3}\Sigma_0 \left(\frac{\bar{\varepsilon}_{\rm e}}{\mathscr{E}_0}\right)^{(1-n)/n} \frac{\bar{\varepsilon}_{ij}}{\mathscr{E}_0} \left\{1 + \rho\left(\frac{n+1}{n}\right) f_{\rm p}\right\}. \tag{78b}$$

The above derivation of the strengthening due to a dilute concentration of particles is based on the macroscopic potential Φ. An equivalent approach is to define a macroscopic strain energy density W of the heterogeneous body in terms of the strain energy density w at each point of the matrix defined in (61), giving

$$VW(\bar{\boldsymbol{\varepsilon}}, \bar{\boldsymbol{\chi}}) \equiv \int_{V_{\rm m}} w(\boldsymbol{\varepsilon}, \boldsymbol{\chi})\, {\rm d}V. \tag{79}$$

With the aid of relation (63) this may be re-written as

$$W(\bar{\boldsymbol{\varepsilon}}, \bar{\boldsymbol{\chi}}) = w(\bar{\boldsymbol{\varepsilon}}, \bar{\boldsymbol{\chi}}) + \rho W_{\rm I}(\bar{\boldsymbol{\varepsilon}}, \bar{\boldsymbol{\chi}}), \tag{80}$$

where W_{I} is interpreted as the change in the strain energy density due to the presence of a rigid particle of unit volume in an infinite matrix medium subjected to the remote strain state $(\bar{\boldsymbol{\varepsilon}}, \bar{\boldsymbol{\chi}})$. The macroscopic average stress $\bar{\mathbf{s}}$ is related to the macroscopic strain $\bar{\varepsilon}$ by

$$\bar{s}_{ij} = \frac{\partial W(\bar{\boldsymbol{\varepsilon}}, \bar{\boldsymbol{\chi}})}{\partial \bar{\varepsilon}_{ij}}. \tag{81}$$

Finally, identification of W_{I} from expressions (78)–(81) gives

$$W_{\mathrm{I}} = \Sigma_0 \mathscr{E}_0 \left(\frac{\bar{\varepsilon}_{\mathrm{e}}}{\mathscr{E}_0} \right)^{(n+1)/n} f_{\mathrm{p}} \tag{82}$$

and a comparison of (77) and (82) shows that $W_{\mathrm{I}}(\bar{\boldsymbol{\varepsilon}}, \bar{\boldsymbol{\chi}}) \equiv -\Phi_{\mathrm{I}}(\bar{\boldsymbol{\sigma}}, \bar{\mathbf{m}})$.

5.1. *Isolated rigid particle in an infinite matrix*

The change in potential Φ_{I} due to the presence of an isolated rigid particle in an infinite matrix is calculated using the numerical procedure outlined in Appendix A. In brief, the boundary value problem is an isolated rigid particle in the power law matrix (13), with remote axisymmetric loading

$$\sigma_{33}^{\infty} = S; \quad \sigma_{11}^{\infty} = \sigma_{22}^{\infty} = T, \tag{83}$$

using the notation introduced in Fig. 3. Since the body is incompressible the response is independent of the mean stress $\sigma_{\mathrm{m}}^{\infty} = \frac{1}{3}S + \frac{2}{3}T$, and without loss of generality we put $\sigma_{\mathrm{m}}^{\infty} = 0$. The remote effective stress $\sigma_{\mathrm{e}}^{\infty}$ is given by $\sigma_{\mathrm{e}}^{\infty} = |S - T|$, and is taken to be the measure of remote stress on the body. The boundary value problem is solved by minimizing the potential energy functional (74) using a Rayleigh–Ritz procedure. This problem has been considered previously by DUVA (1984) for a conventional power law deformation theory solid; for the conventional solid the appropriate boundary condition is a vanishing displacement on the surface of the bonded rigid particle. The solution for the more general couple stress solid approaches that of the conventional solid in the limit $l \to 0$. In fact, the results in the current study for $l = 0$ are more accurate than those presented by DUVA (1984) since more terms were taken in the series expansion of the minimization procedure.

The result (78b) for the strengthening due to a dilute concentration of particles reduces in the case of uniaxial tension $(\bar{\sigma}, \bar{\varepsilon})$ to

$$\bar{\sigma} = \Sigma_0 \left\{ 1 + \rho \left(\frac{n+1}{n} \right) f_{\mathrm{p}} \right\} \left(\frac{\bar{\varepsilon}}{\mathscr{E}_0} \right)^{1/n}. \tag{84}$$

Thus the factor $\rho(n+1)f_{\mathrm{p}}/n$ is the relative strengthening due to the particles at a given strain $\bar{\varepsilon}$. Calculated values of f_{p} are shown in Fig. 4 as a function of l/a for selected values of n in the range 1–10. Recall that f_{p} gives a direct non-dimensional measure of the macroscopic strengthening due to a dilute concentration of the rigid particles. It is noted from Fig. 4 that f_{p} increases dramatically with increasing l/a. For example, at $n = 5$, f_{p} increases from 0.80 at $l/a = 0$ to 4.82 at $l/a = 1$. The role played by the

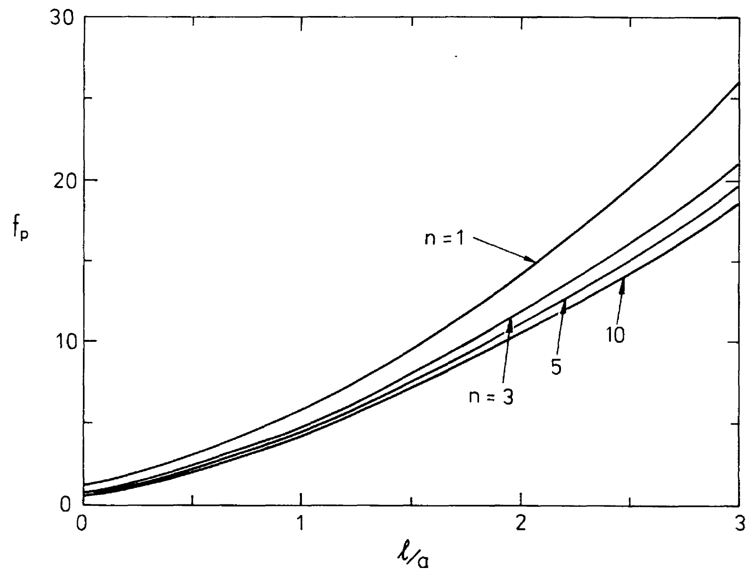

FIG. 4. The strengthening of a power law couple stress solid due to bonded rigid particles of radius a.

strain hardening index n is more modest, with a small decrease in f_p with increasing n.

The numerical results for f_p are well approximated by the formula

$$\frac{f_p}{f_0} = \left(\frac{a+\alpha(n)l}{a}\right)^3 \tag{85}$$

for $0 \leqslant l/a \leqslant 0.5$. Here, f_0 is the value of f_p at $l/a = 0$, and $\alpha(n)$ is a numerical coefficient obtained by curve fitting (85) to the numerical results over $0 \leqslant l/a \leqslant 0.5$. Values for f_0 and α are listed in Table 1 over a range in values for n. The formula (85) is compared with the numerical results in Fig. 5: the agreement is adequate for our purposes.

TABLE 1. *Curve fitted parameters* f_0 *and* α *in* (85) *for the strengthening* f_p *due to an isolated rigid particle*

n	f_0	α
1	1.25	0.693
3	0.800	0.904
5	0.666	1.02
10	0.546	1.16

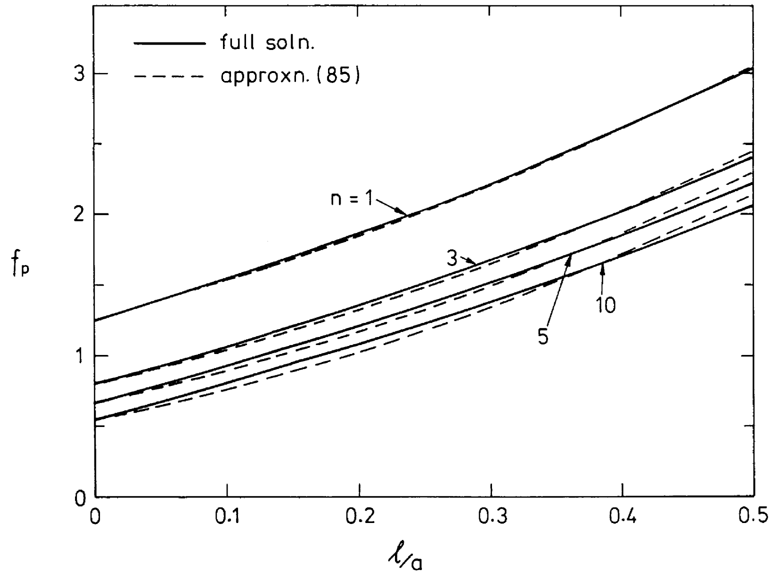

FIG. 5. Comparison of approximate formula (85) for the strengthening due to bonded, rigid particles with the full numerical solution.

We interpret (85) in the following manner: a particle of radius a in the couple stress medium is equivalent to a particle of radius $a+\alpha l$ in the conventional J_2 deformation theory solid insofar as strengthening is concerned. This is consistent with the notion that a boundary layer of thickness αl surrounds the rigid particle in the couple stress medium. The case of a boundary layer in a linear solid under remote simple shear has already been discussed in Section 2.2. In that case the exponential decay length of the boundary layer was $(1/\sqrt{2})l = 0.71l$, which is in close agreement with the fitted value $\alpha(n=1)l = 0.69l$ in the particle problem. The formula (85) suggests a remarkably strong size effect for the bonded rigid inclusion.

It is emphasized that the higher order strain gradient theory demands an extra boundary condition at the surface of the bonded rigid particle in addition to the standard boundary condition $\mathbf{u} = \mathbf{0}$. The most appropriate additional boundary condition remains an open issue; it may be that our choice of $\boldsymbol{\theta} = \mathbf{0}$ on the surface of the particle results in a greater size effect than some other choice. It should also be borne in mind that the strain gradient plasticity law we have employed loses its physical basis in the limit $a/l \to 0$. In that limit, plastic flow becomes controlled by individual dislocation–particle interactions rather than the interaction of a large number of dislocations with a single obstacle.

The nature of the strain field in the vicinity of the rigid particle is explored further in Fig. 6 for the case $n = 5$. The variation of the strain measures ε_e, χ_e and $\mathscr{E}$ with polar angular coordinate ω around the particle is shown in Figs 6(a)–(c) at a fixed radius $r = 1.1a$, and for a range of values of l/a. In the absence of couple stresses $(l/a = 0)$, a local amplification exists in ε_e and χ_e due to the presence of the particle; ε_e peaks at $\omega \approx 30^\circ$ and χ_e peaks at $\omega \approx 45^\circ$. The amplification is fairly local to the particle: this is demonstrated by the rapid drop-off in χ_e with increasing radius r for the case $l/a = 0$ as shown in Fig. 6(d). Nevertheless, it is the presence of the large values of χ_e near the particle which leads to local strengthening around the particle,

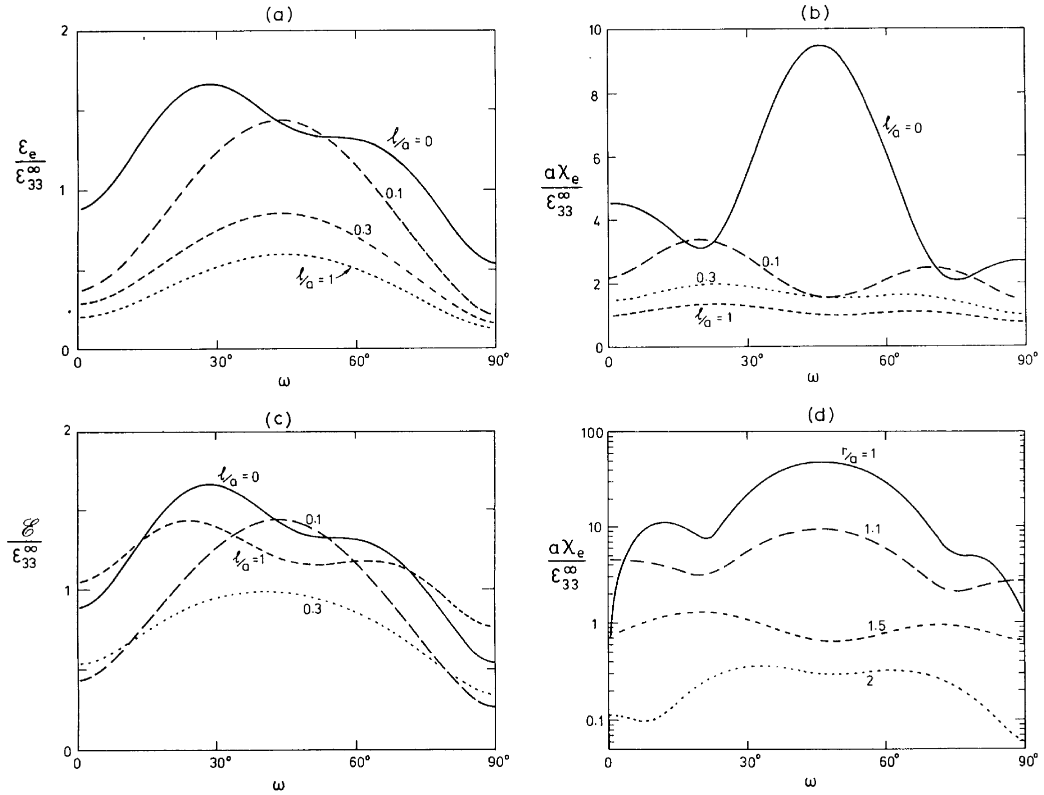

FIG. 6. Strain field near a bonded rigid particle, for $n = 5$ and a range of values of l/a. Plot of (a) effective strain ε_e, (b) effective curvature χ_e and (c) overall strain measure $\mathscr{E}$ versus polar coordinate ω, all for a radius $r = 1.1a$. (d) Distribution of effective curvature χ_e around particle for a range of fixed radii, and for vanishing l/a.

and to enhanced macroscopic strengthening when the constitutive law includes a strain gradient term.

When l/a is finite, the boundary condition $\mathbf{u} = \boldsymbol{\theta} = \mathbf{0}$ on the surface of the rigid particle implies that ε_e vanishes there. Near to the particle boundary and within the matrix the magnitude of both ε_e and χ_e diminishes with increasing l/a as illustrated in Figs 6(a) and (b). The overall strain measure $\mathscr{E} \equiv \sqrt{\varepsilon_e^2 + l^2\chi_e^2}$ varies in a complicated manner with l/a, see Fig. 6(c). As l/a is increased from zero, the magnitude of $\mathscr{E}$ changes from the limiting value of $\mathscr{E} = \varepsilon_e$ to $\mathscr{E} \approx l\chi_e$ at large l/a. At a fixed radius $r = 1.1a$, $\mathscr{E}$ exhibits a minimum for $l/a \approx 0.3$, over most of the range of θ. This behaviour is qualitatively consistent with that exhibited in the shear boundary layer shown in Fig. 2(b) for the linear solid; there $\mathscr{E}$ shows a minimum at a distance of approximately $0.5l$ from the rigid boundary.

6. Softening Due to a Dilute Concentration of Spherical Voids

Now consider a dilute concentration of traction free voids in a power law matrix characterized by (13). The voided solid is subjected to an average stress $\bar{\boldsymbol{\sigma}}$ and strain gradients are taken to be sufficiently mild at the macroscopic level that the average couple stress $\bar{\mathbf{m}}$ (and $\bar{\tau}$) vanishes. The voids are taken to be spherical in shape and of uniform radius a. The macroscopic response is given by (64) and (65) in terms of the change in potential Φ_I due to introduction of an isolated void in an infinite matrix subjected to the remote uniform stress $\bar{\boldsymbol{\sigma}}$. By isotropy, Φ_I depends at most upon the three invariants of $\bar{\boldsymbol{\sigma}}$: the mean stress $\bar{\sigma}_m \equiv \frac{1}{3}\bar{\sigma}_{kk}$, the von Mises effective stress $\bar{\sigma}_e \equiv \sqrt{\frac{3}{2}\bar{s}_{ij}\bar{s}_{ij}}$ and the third invariant $J_3 \equiv (\bar{s}_{ij}\bar{s}_{jk}\bar{s}_{ki})^{1/3}$. We argue that the dependence of Φ_I upon the third invariant is relatively minor compared to its dependence on $\bar{\sigma}_m$ and $\bar{\sigma}_e$. Then, because Φ_I is homogeneous of degree $n+1$ in $\boldsymbol{\sigma}$, one can write

$$\Phi_I = \Sigma_0 \mathscr{E}_0 \left(\frac{\bar{\sigma}_e}{\Sigma_0}\right)^{n+1} f_v(X, l/a, n), \tag{86}$$

where

$$X = \bar{\sigma}_m/\bar{\sigma}_e. \tag{87}$$

The kernel problem of an isolated void in an infinite matrix is required in order to determine values for the non-dimensional function f_v. The function f_v provides a convenient measure of the softening due to the presence of voids.

The macroscopic strain $\bar{\varepsilon}$ is given in terms of the macroscopic stress $\bar{\boldsymbol{\sigma}}$ via (64), (66) and (86) as

$$\bar{\varepsilon}_{ij} = \tfrac{3}{2}\mathscr{E}_0\left(\frac{\bar{\sigma}_e}{\Sigma_0}\right)^{n-1}\frac{\bar{s}_{ij}}{\Sigma_0} + \rho\mathscr{E}_0\left\{\frac{3}{2}\left(\frac{\bar{\sigma}_e}{\Sigma_0}\right)^{n-1}\frac{\bar{s}_{ij}}{\Sigma_0}\left[(n+1)f_v - X\frac{\partial f_v}{\partial X}\right] + \frac{1}{3}\left(\frac{\bar{\sigma}_e}{\Sigma_0}\right)^{n}\frac{\partial f_v}{\partial X}\delta_{ij}\right\}, \tag{88}$$

in agreement with the formula (1.11) in DUVA and HUTCHINSON (1984). Unlike the case of the dilute concentration of rigid particles, (88) cannot be inverted in an explicit manner; the results (79) and (80) still hold and $W_I(\bar{\varepsilon}, \bar{\chi}) = -\Phi_I(\bar{\boldsymbol{\sigma}}, \bar{\mathbf{m}})$.

6.1. *Isolated void in an infinite matrix*

Accurate numerical solutions have been obtained for the kernel problem of an isolated spherical void of radius a in an infinite power law matrix obeying the constitutive relations (8)–(13). The matrix is subjected to the remote axisymmetric loading (S, T) as defined in (83). The solution procedure is based on minimization of the potential energy functional (74), and the use of (76) and (86) in order to extract the values of f_v. Further details on the numerical procedure are given in Appendix B.

For axisymmetric remote stress states, all dependence of Φ_v on $\boldsymbol{\sigma}^\infty$ can be expressed using the remote mean stress $\sigma_m^\infty \equiv \frac{1}{3}S + \frac{2}{3}T$ and $\sigma^\infty \equiv S - T$, since they are linear combinations of S and T. Furthermore, because Φ_v is even and homogeneous of degree $(n+1)$ in $\boldsymbol{\sigma}^\infty$, Φ_v can be written quite generally for axisymmetric loading as

$$\Phi_v(\boldsymbol{\sigma}^\infty) = \Sigma_0\mathscr{E}_0\left(\frac{\sigma_e^\infty}{\Sigma_0}\right)^{n+1} f_v(x, l/a, n), \tag{89}$$

where $x \equiv \sigma_m^\infty/\sigma^\infty$ and the remote effective stress $\sigma_e^\infty = |\sigma^\infty|$. The distinction between

x in (89) and X in (86) is deliberate. Only if $\Phi_v(\boldsymbol{\sigma}^\infty)$ is independent of the third stress invariant $J_3 = (s_{ij}^\infty s_{jk}^\infty s_{ki}^\infty)^{1/3}$ is it true that expressions (86) and (89) for $\Phi_v(\boldsymbol{\sigma}^\infty)$ coincide. We shall find that (89) is almost even in x, and we thereby argue that (86) is a good approximation to (89) for axisymmetric stress states, and indeed is accurate for general stress states.

The function $f_v(x, l/a, n)$ is a non-dimensional measure of the softening due to the presence of a dilute concentration of voids; f_v is plotted against x in Fig. 7 for selected values of l/a in the range of 0–3, and $n = 5$. It is clear that f_v decreases slightly with increasing l/a: in other words, the degree of macroscopic softening decreases with decreasing void size. The effect is not a large one, however. For example, for $x = 0$, f_v decreases from 1.29 to 0.848 when l/a is increased from 0 to 3. Note further from Fig. 7 that numerical results for f_v are almost even in x. For $S > T$, x is positive and $X = x$. For $S < T$, x is negative and $X = -x$. Only if f_v were even in x would the representation (86) reproduce exactly the axisymmetric results.

Results for the limit of vanishing l/a coincide with those given by DUVA and HUTCHINSON (1984). Also, their low and high triaxiality approximations carry over to the current case with only slight modification due to l/a, as will now be discussed.

Low triaxiality approximation. For any given n and l/a, f_v is well approximated by the quadratic formula

$$f_v(x, l/a, n) = f^*(l/a, n) + \tfrac{1}{2}\kappa(l/a, n)x^2, \tag{90}$$

where f^* equals the value of f_v at $x = 0$ and the curvature κ is evaluated by fitting (90) to the mean of the values of f_v at $x = \pm 1$. Results for f^* and κ are listed in Table

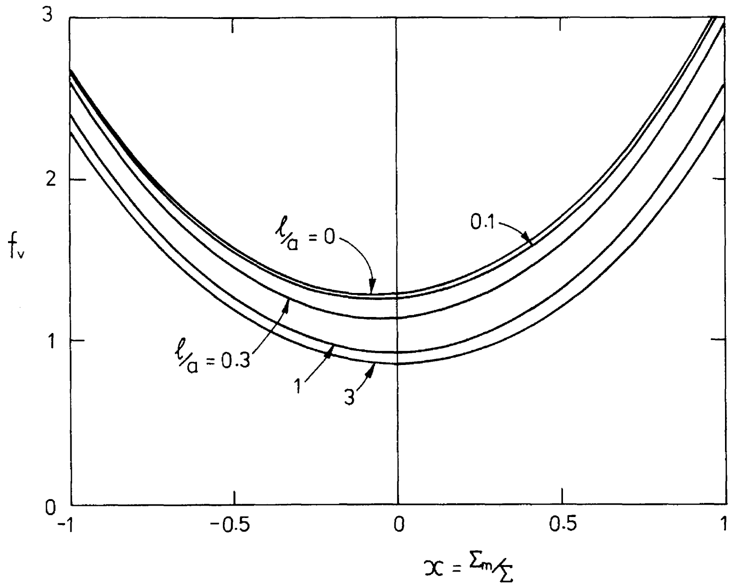

FIG. 7. The softening due to a dilute concentration of spherical voids: $f_v(x, l/a, n)$ for axisymmetric remote stressing in the low triaxiality range, $n = 5$.

TABLE 2. *Curve fitted parameters* f* *and* κ *in the low triaxiality approximation* (90) *for the softening* f_v *due to an isolated void*

	$n = 3$		$n = 5$	
l/a	f^*	κ	f^*	κ
0	1.17	2.93	1.29	3.23
0.1	1.15	2.95	2.16	3.26
0.3	1.04	3.00	1.14	3.30
1	0.837	2.94	0.924	3.17
3	0.764	2.83	0.848	3.01

2, and are plotted against l/a in Fig. 8 for $n = 3$ and 5. It is clear that κ varies little with l/a or with n. There is a relatively small increase in f^* with decreasing l/a and with increasing n.

The dilatation ΔV of the void is related to f_v by

$$\frac{\Delta V}{V} = \frac{\partial \Phi_v}{\partial \sigma_m^\infty} = \mathscr{E}_0 \left(\frac{\sigma_e^\infty}{\Sigma_0} \right)^{n-1} \frac{\sigma^\infty}{\Sigma_0} \frac{\partial f_v}{\partial x}. \tag{91}$$

On noting that the remote strain is given by

$$\varepsilon_{33}^\infty = \mathscr{E}_0 \left(\frac{\sigma_e^\infty}{\Sigma_0} \right)^{n-1} \frac{\sigma^\infty}{\Sigma_0}, \tag{92}$$

the normalized dilatation of the void is

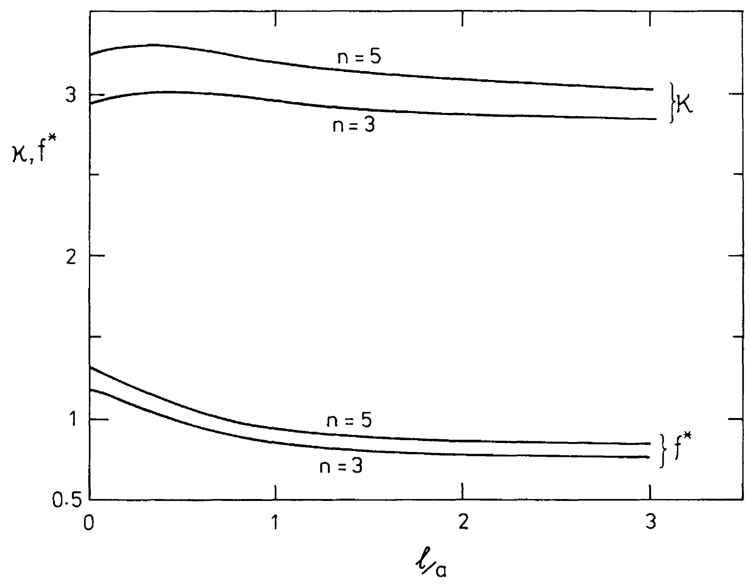

FIG. 8. Effect of size ratio l/a upon the curve fitted parameters f^* and κ in the low triaxiality approximation (89).

$$\frac{\Delta V}{\varepsilon_{33}^{\infty} V} = \frac{\partial f_{\mathrm{v}}(x, l/a, n)}{\partial x}. \tag{93}$$

Thus, a low triaxiality approximation for the normalized void dilatation is given by differentiation of (90) with respect to x. The resulting approximation is in good agreement with the full numerical solution, as illustrated in Fig. 9(a). Plainly, the normalized dilatation is not sensitive to the normalized void size, a/l.

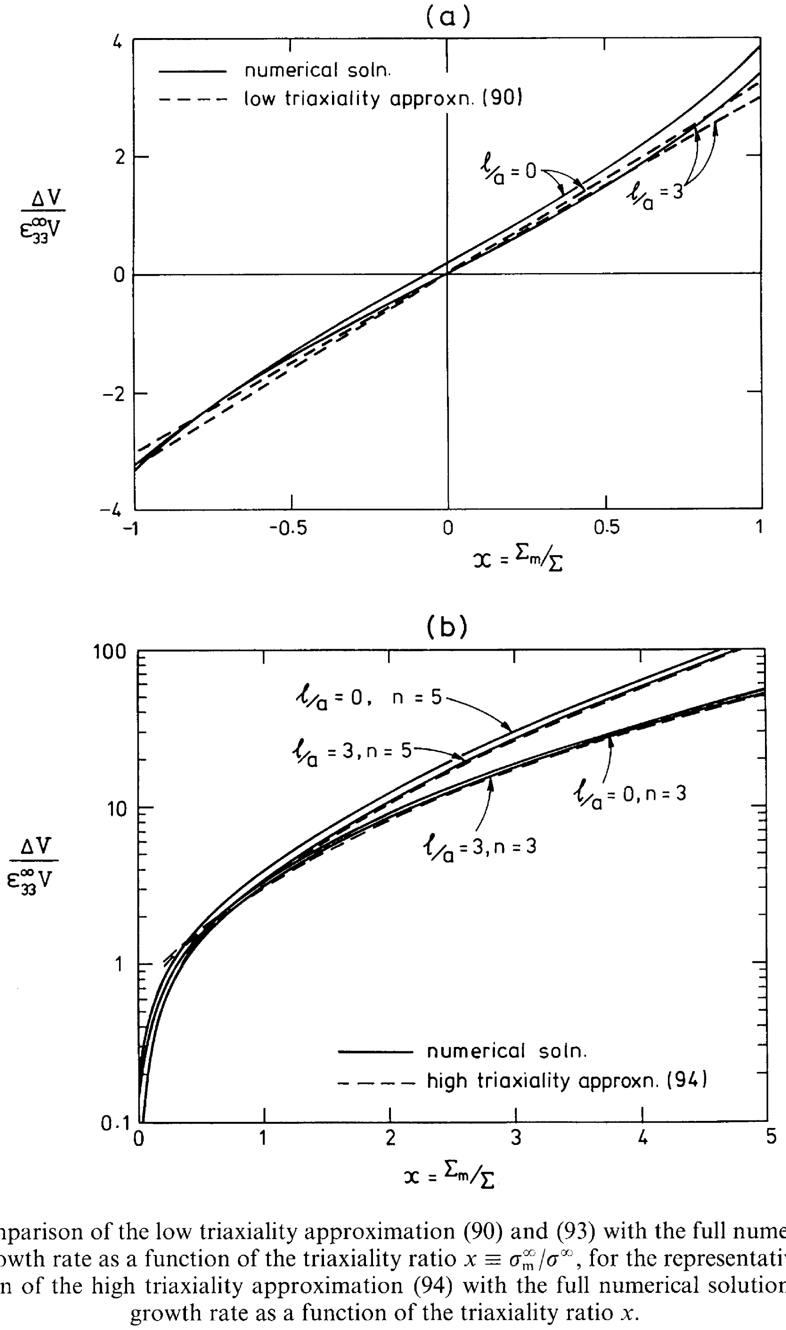

FIG. 9. (a) Comparison of the low triaxiality approximation (90) and (93) with the full numerical solution for the void growth rate as a function of the triaxiality ratio $x \equiv \sigma_{\mathrm{m}}^{\infty}/\sigma^{\infty}$, for the representative case $n = 5$. (b) Comparison of the high triaxiality approximation (94) with the full numerical solution for the void growth rate as a function of the triaxiality ratio x.

High triaxiality approximation. BUDIANSKY *et al.* (1982) derived a high triaxiality approximation for the normalized dilatation

$$\frac{\Delta V}{\varepsilon_{33}^{\infty} V} = \tfrac{3}{2} m \left\{ \frac{3|x|}{2n} - G(n,m) \right\}^{n} \tag{94a}$$

where, using their notation,

$$-G(n,m) = (n-1)\left[\frac{n+g(m)}{n^2}\right] \tag{94b}$$

and $m \equiv x/|x|$, $g(1) = 0.4319$ and $g(-1) = 0.4031$.

The high triaxiality approximation was obtained by neglecting all but the spherically symmetric contribution to the displacement field in the minimization procedure. The spherically symmetric field dominates at moderate to high triaxialities, and involves only uniaxial straining of each material element. It introduces no curvature χ to any material point surrounding the void and so the high triaxiality approximation remains valid in the couple stress formulation. This is the underlying reason why no strong size effect is noted for the void problem. A comparison of the high triaxiality approximation with the full numerical solution is given in Fig. 9(b). At high triaxialities, the dependence of f_v and $\Delta V/\varepsilon_{33}^{\infty} V$ upon l/a disappears, and the approximation (94) is an accurate representation for all l/a.

Evolution of void shape. BUDIANSKY *et al.* (1982) found that the shape to which a spherical void evolves is sensitive to the value of n. For S slightly greater than T (and both positive), the void grows towards a prolate shape for $n < 2$, and towards an oblate shape for $n > 2$.

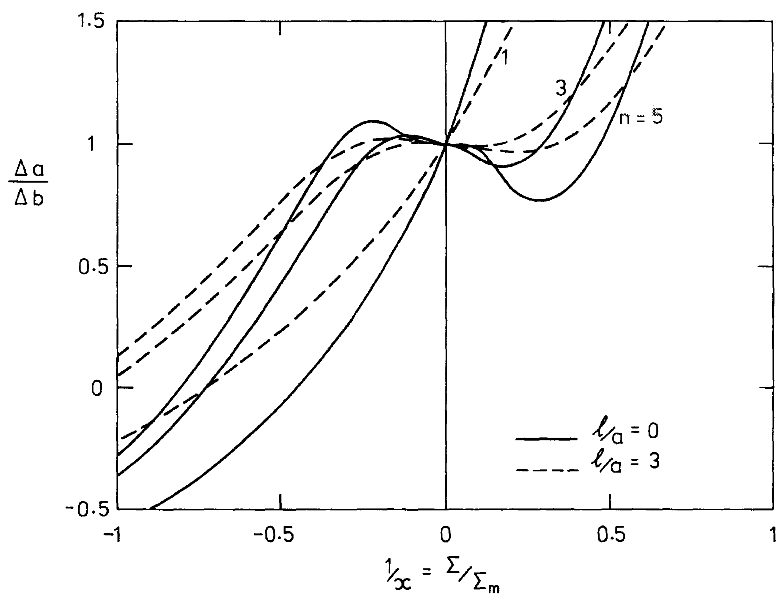

FIG. 10. Initial shape change of spherical void.

Let Δa be the increment in growth of the void from its centre to its pole, and Δb be the corresponding increment in the radius of the equator. The change in void shape as described by $\Delta a/\Delta b$ is plotted against $\sigma^{\infty}/\sigma_{\mathrm{m}}^{\infty} = 3(S-T)/(S+2T) = 1/x$ in Fig. 10, for n in the range 1–5. The results for a range of values of l/a confirm the previous finding of BUDIANSKY *et al.* (1982) for vanishing l/a: for $n > 2$ there exists a limited range of $\sigma^{\infty}/\sigma_{\mathrm{m}}^{\infty} > 0$ over which the void grows towards an oblate shape. Similarly, for $n > 2$, there exists a limited range of $\sigma^{\infty}/\sigma_{\mathrm{m}}^{\infty} < 0$ over which the void adopts a prolate shape. The phenomenon is moderated by the presence of strain gradient hardening: the reversal in slope of the $\Delta a/\Delta b$ versus $\sigma^{\infty}/\sigma_{\mathrm{m}}^{\infty}$ curve becomes less severe with increasing l/a.

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APPENDIX A: NUMERICAL ANALYSIS OF ISOLATED RIGID PARTICLE

The kernel problem for the macroscopic strengthening due to a dilute concentration of rigid particles consists of an isolated rigid spherical particle of radius a, bonded to an infinite matrix under remote axisymmetric loading $S = \frac{2}{3}\Sigma$, $T = -\frac{1}{3}\Sigma$, as shown in Fig. 3. The matrix obeys the strain gradient version of J_2 deformation theory outlined in Section 2, and the power law relation (13) is adopted.

The boundary value problem is solved by a Rayleigh–Ritz minimization of the potential energy functional (74) in a similar manner to that outlined by BUDIANSKY *et al.* (1982). Application of the principle of virtual work to (74), together with (63) and the boundary condition $\mathbf{u} = \mathbf{0}$ on the particle surface enables (74) to be simplified to

$$P_{\rm I}(\mathbf{u}) = \int_{V_{\rm m}} [w(\boldsymbol{\varepsilon}, \boldsymbol{\chi}) - w(\boldsymbol{\varepsilon}^{\infty}, \boldsymbol{\chi}^{\infty} = \mathbf{0}) - s_{ij}^{\infty}\,\tilde{\varepsilon}_{ij}]\,\mathrm{d}V + V_{\rm I}\phi(\boldsymbol{\sigma}^{\infty}, \mathbf{m}^{\infty} = \mathbf{0}). \tag{A1}$$

In order to minimize $P_{\rm I}$ the displacement field $\mathbf{u}$ is written as a series expansion in the spherical polar co-ordinates (r, ω) shown in Fig. 3. By incompressibility the displacement field (u_r, u_ω) satisfies

$$u_{r,r} + 2\frac{u_r}{r} + \frac{u_\omega}{r}\cot\omega + \frac{u_{\omega,\omega}}{r} = 0 \tag{A2}$$

and so the displacement components (u_r, u_ω) may be expressed in terms of the stream function $\psi(r, \omega)$ as

$$u_r = -\frac{1}{r^2 \sin\omega}(\psi \sin\omega)_{,\omega}, \quad u_\omega = \frac{1}{r}\psi_{,r}. \tag{A3}$$

In the absence of the rigid particle the stream function is

$$\psi = \psi^\infty(r, \omega) = \frac{\varepsilon_{33}^{\infty}}{6} r^3 P_{2,\omega}(\cos\omega), \tag{A4}$$

in terms of the Legendre function $P_2(\cos\omega)$. The presence of the bonded rigid particle modifies ψ to

$$\psi = \psi^\infty + \tilde{\psi}^\infty + \tilde{\psi}, \tag{A5}$$

where $\tilde{\psi}^\infty$ is chosen to ensure that both $\mathbf{u}$ and $\boldsymbol{\theta}$ associated with the primary field $(\psi^\infty + \tilde{\psi}^\infty)$ vanish at the surface of the particle, giving

$$\tilde{\psi}^\infty(r, \omega) = \frac{\varepsilon_{33}^{\infty}}{6} P_{2,\omega}(\cos\omega)\left[-10a^3 + \frac{15a^4}{r} - \frac{6a^5}{r^2}\right]. \tag{A6}$$

The secondary field $\tilde{\psi}$ contains free amplitude factors in order to minimize $P_{\rm I}(\mathbf{u})$ in (A1). A series expansion of the stream function, with due regard to the symmetries of the problem, leads to

$$\tilde{\psi}(r, \omega) = \sum_{j=2,4,6,\ldots}^{J} \sum_{m=0,1,2,\ldots}^{M} a_{jm}\tilde{\psi}_{jm}(r, \omega), \tag{A7}$$

where

$$\tilde{\psi}_{jm} = P_{j,\omega}(\cos\omega)[r^{-m} + Ar^{-(m+1)} + Br^{-(m+2)} + Cr^{-(m+3)}]. \quad (A8)$$

The parameters A, B and C are chosen to ensure that the contribution to $\mathbf{u}$ and $\boldsymbol{\theta}$ associated with each term $\tilde{\psi}_{jm}$ vanishes on the surface of the particle, $r = a$. This implies that

$$A = -3a, \quad B = 3a^2, \quad C = -a^3. \quad (A9)$$

For convenience, the amplitude factors a_{jm} are denoted collectively by $\{A_k\}$, and these are the variables with respect to which P_{I} is minimized. Explicit expressions for both P_{I} and the gradient $\{\partial P_{\mathrm{I}}/\partial A_k\}$ are used in the Newton–Raphson minimization procedure. By the change of variable $\mu = a/r$, the r-integration in the volume integral of (A1) is converted to an integration over the range $0 \leqslant \mu \leqslant 1$. A 10 point Gaussian quadrature formula is used for integration with respect to μ and with respect to ω; due to the symmetry of the problem the integration interval for ω is from 0 to $\pi/2$. The number of terms in the series expansion (A7) is varied in order to obtain adequate accuracy: by setting $J = 6$ and $M = 4$ in (A7) results are accurate to within 0.5%.

Appendix B: Numerical Analysis of Isolated Void

The numerical analysis for an isolated void in a power law matrix (13) follows closely that given in Appendix A for the rigid particle. With remote axisymmetric stressing S and T, as shown in Fig. 3, the potential energy functional (74) is rewritten as

$$P_{\mathrm{I}}(\mathbf{u}) = \int_{V_{\mathrm{m}}} [w(\boldsymbol{\varepsilon}, \boldsymbol{\chi}) - w(\boldsymbol{\varepsilon}^{\infty}, \boldsymbol{\chi}^{\infty} = \mathbf{0}) - s_{ij}^{\infty}\tilde{\varepsilon}_{ij}]\,\mathrm{d}V - \int_{A_{\mathrm{I}}} [\sigma_{ij}^{\infty} n_i \tilde{u}_j]\,\mathrm{d}A - V_{\mathrm{I}} w(\boldsymbol{\varepsilon}^{\infty}, \boldsymbol{\chi}^{\infty} = \mathbf{0}). \quad (B1)$$

Since the matrix in incompressible (A2) holds and the displacements u_r and u_ω are again expressed in terms of a stream function $\psi(r, \omega)$ by (A3), where now we take $\psi(r, \omega)$ to be

$$\psi(r, \omega) = \frac{\varepsilon_{33}^{\infty}}{6} r^3 P_{2,\omega}(\cos\omega) + A_0 \cot\omega + \sum_{j=2,4,6,\ldots}^{J} \sum_{m=0,1,2,\ldots}^{M} a_{jm} r^{-m} P_{j,\omega}(\cos\omega). \quad (B2)$$

The lead term in (B2) is the remote field; the second term is the spherically symmetric contribution and scales with the free amplitude factor A_0. The remaining terms scale with the amplitude factors a_{jm}. In contrast to the case of the bonded rigid particle no additional constraints are needed to enforce the appropriate boundary condition on the surface of the void: minimization of P_{I} ensures satisfaction of the natural boundary condition that the void is traction free.

The reduction of the minimization problem to a standard algebraic problem for the unknown amplitude factors $\{A_0, a_{jm}\}$ follows the minimization procedure given in Appendix A, and explained in more detail by Budiansky *et al.* (1982). A total of 16 free amplitude factors were taken for $\{A_0, a_{jm}\}$ with $J = 6$ and $M = 4$ in (B2); this gives a solution which is accurate to within about 0.1%.