# Spherical Cavity Expansion in a Drucker-Prager Solid

by

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**Abstract**

A finite strain analysis is presented for the pressurized spherical cavity embedded in a Drucker-Prager medium. Material behaviour is modelled by a non-associated deformation theory with strain-hardening. The governing equations are reduced to a single differential equation with the effective stress as the independent variable. Some related topics are discussed including the elastic/perfectly-plastic solid and the cavitation limit. Numerical examples illustrate the pressure sensitivity of material response.

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## Introduction

It is commonly accepted that the plastic response of soils is pressure (mean stress) dependent. Drucker and Prager (1952) have proposed a simple yield function which models pressure sensitivity of perfectly-plastic materials by adding a linear pressure term to the von-Mises yield stress. In the present paper we study the spherical cavity problem employing a more general version of the Drucker-Prager solid. Material behaviour is assumed to be elastoplastic with arbitrary strain hardening, and the plastic flow potential is not identical with the yield surface. Thus, the flow rule is non associated.

The constitutive model, described in the next section, contains a complete 3D derivation of the appropriate incremental stress-strain relations with large deformations. For the pressurized cavity, however, where the stress path is nearly proportional, we introduce a related deformation theory. It is shown that the governing equations can then be reduced to a single first order non-linear differential equation with the effective stress as the independent variable (Durban and Baruch, 1976). For the associated case the flow theory coincides exactly with the deformation theory.

Some special cases are discussed in detail with emphasis on the elastic/perfectly-plastic material. We show, in particular, that both flow and deformation theories predict identical stresses and strains, even when the flow rule is non associated.

When the internal pressure approaches a certain asymptotic value the cavity will expand spontaneously in a self similar mode. This cavitation limit is examined here for both flow and deformation theories. The paper concludes with a few numerical examples that illustrate the pressure sensitivity of material response.

## Constitutive Model

The Drucker-Prager (1952) theory suggests a modification of the von-Mises yield function by accounting for pressure dependence of the yield surface. The Drucker-Prager yield surface $\phi(\underset{\sim}{\sigma})$ is defined by

$$\phi = \left(\frac{3}{2}\underset{\sim}{S}\cdot\cdot\underset{\sim}{S}\right)^{1/2} + \frac{1}{3}\mu(\underset{\sim}{\sigma}\cdot\cdot\underset{\sim}{I}) - \sigma_f = 0 \tag{1a}$$

where $\underset{\sim}{\sigma}$ is the Cauchy stress tensor, $\underset{\sim}{S} = \underset{\sim}{\sigma} - \frac{1}{3}\underset{\sim}{\sigma} \cdot\cdot \underset{\sim}{I}\,\underset{\sim}{I}$ is the stress deviator, $\underset{\sim}{I}$ - the second order unit tensor, $\mu$ - a material parameter and $\sigma_f$ is the stress which measures the size of the yield surface. We may rewrite (1a) as

$$\phi = \sigma_e - \sigma_f \tag{1b}$$

where we define the effective stress as

$$\sigma_e = \left(\frac{3}{2}\underset{\sim}{S} \cdot\cdot \underset{\sim}{S}\right)^{1/2} + \frac{1}{3}\mu(\underset{\sim}{\sigma} \cdot\cdot \underset{\sim}{I})$$

While in their 1952 paper Drucker and Prager assumed perfectly plastic behaviour along with an associated flow rule, we shall allow here for strain hardening and for a plastic potential g which is similar to but does not coincide with the yield surface.

We define g as

$$g = \left(\frac{3}{2}\underset{\sim}{S} \cdot\cdot \underset{\sim}{S}\right)^{1/2} + \frac{1}{3}\eta(\underset{\sim}{\sigma} \cdot\cdot \underset{\sim}{I}) \tag{2}$$

where $\eta \neq \mu$ , and the plastic branch of the strain rate tensor is

$$\underset{\sim}{D}^P = \Lambda \frac{dg}{d\underset{\sim}{\sigma}} = \Lambda\left[\left(\frac{3}{2}\underset{\sim}{S} \cdot\cdot \underset{\sim}{S}\right)^{-1/2}\left(\frac{3}{2}\underset{\sim}{S}\right) + \frac{1}{3}\eta\underset{\sim}{I}\right] \tag{3}$$

with $\Lambda$ standing for the proportionality factor. Invoking now the principle of plastic power equivalence, namely

$$\underset{\sim}{\sigma} \cdot\cdot \underset{\sim}{D}^P = \sigma_f \dot{\varepsilon}_P \tag{4}$$

where the total plastic strain $\varepsilon_P$ is a known function of $\sigma_f$, we find with the aid of (3) that

$$\Lambda = \frac{\sigma_f}{g}\dot{\varepsilon}_P \tag{5}$$

The plastic strain rate (3) is therefore

$$\underset{\sim}{D}^P = \frac{\sigma_e}{g}\dot{\varepsilon}_P\left[\left(\frac{3}{2}\underset{\sim}{S} \cdot\cdot \underset{\sim}{S}\right)^{-1/2}\left(\frac{3}{2}\underset{\sim}{S}\right) + \frac{1}{3}\eta\underset{\sim}{I}\right] \tag{6}$$

Assuming a standard Hookean relation for the elastic part of the strain rate, viz.

$$\underset{\sim}{D}^E = \frac{1}{2G}\left(\overset{\nabla}{\underset{\sim}{\sigma}} - \frac{\nu}{1+\nu}\overset{\nabla}{\underset{\sim}{\sigma}}\cdot\cdot\underset{\sim}{I}\,\underset{\sim}{I}\right) \tag{7}$$

where $(G,\nu)$ are the usual elastic constants and $\overset{\nabla}{\underset{\sim}{\sigma}}$ denotes the objective Jaumann stress rate, we write the total strain rate as

$$\underset{\sim}{D} = \underset{\sim}{D}^E + \underset{\sim}{D}^P \tag{8}$$

This equation, along with (6) and (7), describes the complete elastoplastic constitutive law for the non-associated Drucker-Prager solid. The time derivative of the effective strain can be expressed in terms of stress and stress rate by

$$\dot{\varepsilon}_P = \frac{d\varepsilon_P}{d\sigma_f}\cdot\frac{d\sigma_f}{d\underset{\sim}{\sigma}}\cdot\cdot\overset{\nabla}{\underset{\sim}{\sigma}} \tag{9}$$

where from (1)

$$\frac{d\sigma_f}{d\underset{\sim}{\sigma}} = \left(\frac{3}{2}\underset{\sim}{S}\cdot\cdot\underset{\sim}{S}\right)^{-1/2}\left(\frac{3}{2}\underset{\sim}{S}\right) + \frac{1}{3}\mu\underset{\sim}{I} \tag{10}$$

It follows that the constitutive relation (8) can be written as

$$\underset{\sim}{D} = \underset{\sim}{M}\cdot\cdot\overset{\nabla}{\underset{\sim}{\sigma}} \tag{11}$$

with the fourth order compliance tensor $\underset{\sim}{M}$ given by

$$\underset{\sim}{M} = \frac{1}{2G}\left(\underset{\sim}{I} - \frac{\nu}{1+\nu}\underset{\sim}{I}\,\underset{\sim}{I}\right) + \frac{\sigma_f}{g}\cdot\frac{d\varepsilon_P}{d\sigma_f}\cdot\frac{dg}{d\underset{\sim}{\sigma}}\cdot\frac{d\sigma_f}{d\underset{\sim}{\sigma}}$$

where $\underset{\sim}{I}$ is the fourth order unit tensor. Notice that, by (3) and (10), the plastic part of $\underset{\sim}{M}$ admits the Cauchy symmetry only for an associated flow rule where $\eta = \mu$ (or, trivially, in hydrostatic loading).

The dependence of $\varepsilon_P$ on the flow stress can be determined from a simple uniaxial compression experiment. Denoting by $\sigma_1$ the compressive stress we have that $\sigma_f = (-1+\frac{1}{3}\mu)\sigma_1$ , $g = (-1+\frac{1}{3}\mu)\sigma_1$ and the constitutive relations specialize to

$$\dot{\varepsilon}_1 = \frac{\dot{\sigma}_1}{E} - \left(1 - \frac{1}{3}\mu\right)\dot{\varepsilon}_P$$

$$\dot{\varepsilon}_2 = \dot{\varepsilon}_3 = -\frac{\nu\dot{\sigma}_1}{E} + \frac{(1 - \frac{1}{3}\mu)(1 + \frac{2}{3}\eta)}{2(1 - \frac{1}{3}\eta)}\dot{\varepsilon}_P \tag{12}$$

where both axial ($\varepsilon_1$) and transverse ($\varepsilon_2$=$\varepsilon_3$) strains are logarithmic. Integrating the first of (12) over the loading history results in the variation of $\varepsilon_P$ with $\sigma_f$. An interesting observation that follows from (12) is that in the deep plastic range $-\varepsilon_2/\varepsilon_1$ approaches asymptotically the value of

$$\frac{\left(1 + \frac{2}{3}\eta\right)}{2\left(1 - \frac{1}{3}\eta\right)} \tag{13}$$

which is independent of $\mu$.

Volume changes are given by

$$\underset{\sim}{I} \cdot\cdot \underset{\sim}{D} = \frac{1 - 2\nu}{E} \underset{\sim}{I} \cdot\cdot \overset{\nabla}{\underset{\sim}{\sigma}} + \eta \frac{\sigma_f}{g} \dot{\varepsilon}_P \tag{14}$$

In pure hydrostatic tension, where $\underset{\sim}{\sigma} = t\,\underset{\sim}{I}$, the plastic dilatation rate becomes equal to $\mu\dot{\varepsilon}_P$ with the flow stress $\sigma_f = \mu t$. The variation of the effective plastic strain with the flow stress, in this particular deformation pattern, is independent of $\eta$.

## The Pressurized Cavity

A spherical cavity, with an initially undeformed radius "a", is embedded in an infinite medium and subjected to increasing internal pressure p. Locating the origin of a spherical polar Lagrangian system (r,θ,φ) at the center of the cavity, we have by symmetry the non vanishing stresses ($\sigma_r$, $\sigma_\theta = \sigma_\varphi$) and strains ($\varepsilon_r$, $\varepsilon_\theta = \varepsilon_\varphi$). The effective stress is

$$\sigma_e = \sigma_\theta - \sigma_r + \frac{1}{3}\mu(\sigma_r + 2\sigma_\theta) \tag{15}$$

since it is expected that under internal pressure $\sigma_\theta > \sigma_r$ at any point within the body.

Similarly, from (2)

$$g = \sigma_\theta - \sigma_r + \frac{1}{3}\eta(\sigma_r + 2\sigma_\theta)$$

$$= \frac{\left(1+\frac{2}{3}\eta\right)\sigma_e + (\eta-\mu)\sigma_r}{\left(1+\frac{2}{3}\mu\right)} \tag{16}$$

Turning to the constitutive relations (8) we have just two scalar equations

$$\dot{\varepsilon}_r = \frac{1}{E}(\dot{\sigma}_r - 2\nu\dot{\sigma}_\theta) + (-1+\frac{1}{3}\eta)\frac{\sigma_e}{g}\dot{\varepsilon}_P \tag{17a}$$

$$\dot{\varepsilon}_\theta = \frac{1}{E}\left[-\nu\dot{\sigma}_r + (1-\nu)\dot{\sigma}_\theta\right] + \left(\frac{1}{2}+\frac{1}{3}\eta\right)\frac{\sigma_e}{g}\dot{\varepsilon}_P \tag{17b}$$

Expecting however a nearly proportional stress path we proceed with the deformation-type theory whereby (17) is replaced by

$$\varepsilon_r = \frac{1}{E}(\sigma_r - 2\nu\sigma_\theta) + \left(-1+\frac{1}{3}\eta\right)\frac{\sigma_e}{g}\varepsilon_P \tag{18a}$$

$$\varepsilon_\theta = \frac{1}{E}\left[-\nu\sigma_r + (1-\nu)\sigma_\theta\right] + \left(\frac{1}{2}+\frac{1}{3}\eta\right)\frac{\sigma_e}{g}\varepsilon_P \tag{18b}$$

Equations (18) are the exact integrals of (17) for the cavity problem when the flow rule is associated ($\eta = \mu \Rightarrow g = \sigma_e$). Otherwise there is an error involved in (18) in comparison with the hon-holonomic model (17). Deviations from proportionality can be examined a-posteriori by evaluating the ratio $\sigma_e/g$. This implies that a sufficient condition for (18) to remain valid in non proportional loading is

$$\frac{|\eta-\mu|}{1+\frac{2}{3}\eta} \cdot \frac{|\sigma_r|}{\sigma_e} << 1 \tag{19}$$

Equilibrium requirements are here governed by the single equation

$$\frac{d\sigma_r}{dr} + 2\frac{1+\frac{du}{dr}}{1+\frac{u}{r}}(\sigma_r - \sigma_\theta) = 0 \tag{20}$$

where u=u(r) is the radial displacement. Recalling however that the finite strain components are

$$\varepsilon_r = \ell n\left(1+\frac{du}{dr}\right) \quad \varepsilon_\theta = \ell n\left(1+\frac{u}{r}\right) \tag{21}$$

we can rewrite (20) in the differential form

$$d\sigma_r = 2(\sigma_\theta - \sigma_r)\exp(\varepsilon_r - \varepsilon_\theta)\frac{dr}{r} \tag{22}$$

Combining this with the strain compatibility relation, from (21),

$$d\varepsilon_\theta = \left[\exp(\varepsilon_r - \varepsilon_\theta) - 1\right]\frac{dr}{r} \tag{23}$$

we arrive at the connection

$$d\sigma_r = \frac{2(\sigma_\theta - \sigma_r)d\varepsilon_\theta}{1 - \exp(\varepsilon_\theta - \varepsilon_r)} \tag{24}$$

which will be reduced to an ordinary differential equation for $\sigma_r$ in terms of the independent variable $\sigma_e$. The formulation centers on the effective stress $\sigma_e$ as the independent variable and is valid for any strain hardening characteristic.

To this end we rearrange (15) to obtain

$$\sigma_\theta - \sigma_r = \frac{\sigma_e - \mu\sigma_r}{(1 + \frac{2}{3}\mu)} \tag{25}$$

Next, from (18),

$$\varepsilon_\theta - \varepsilon_r = \frac{1+\nu}{E}(\sigma_\theta - \sigma_r) + \frac{3}{2}\cdot\frac{\sigma_e}{g}\varepsilon_P \tag{26}$$

or, with the aid of (25),

$$\varepsilon_\theta - \varepsilon_r = \frac{1+\nu}{E}\left(\frac{\sigma_e - \mu\sigma_r}{1 + \frac{2}{3}\mu}\right) + \frac{3}{2}\cdot\frac{\sigma_e}{g}\varepsilon_P \tag{27}$$

Finally, we use (15) to eliminate $\sigma_\theta$ from (18b) resulting in

$$\varepsilon_\theta = \frac{1-\nu}{E}\cdot\frac{\sigma_e}{1+\frac{2}{3}\mu} + \left[(1-2\nu) - \frac{(1-\nu)\mu}{1+\frac{2}{3}\mu}\right]\frac{\sigma_r}{E} + \left(\frac{1}{2} + \frac{1}{3}\eta\right)\frac{\sigma_e}{g}\varepsilon_P \tag{28}$$

The differential form of (28) is

$$d\varepsilon_\theta = \frac{\partial\varepsilon_\theta}{\partial\sigma_e}d\sigma_e + \frac{\partial\varepsilon_\theta}{\partial\sigma_r}d\sigma_r \tag{29}$$

where the partial derivatives are determined from (28) without difficulty.

Substituting now (25), (27) and (29) in (24) we obtain the desired differential equation

$$\frac{d\Sigma_r}{d\Sigma} = -\frac{(\Sigma - \mu\Sigma_r)\left(\alpha_1 + \beta_1 \frac{\Sigma}{G}\varepsilon_P' + \beta_2 \frac{\Sigma_r}{G^2}\varepsilon_P\right)}{\exp\left[\alpha_3(\Sigma - \mu\Sigma_r) + \frac{3}{2}\cdot\frac{\Sigma}{G}\varepsilon_P\right] - 1 + (\Sigma - \mu\Sigma_r)\left(\alpha_2 - \beta_2 \frac{\Sigma}{G^2}\varepsilon_P\right)} \tag{30}$$

where all stress components have been nondimensionalized with respect to the elastic modulus

$$\Sigma_r = \frac{\sigma_r}{E}, \ \Sigma_\theta = \frac{\sigma_\theta}{E}, \ \Sigma = \frac{\sigma_e}{E}, \ \varepsilon_P' = \frac{d\varepsilon_P}{d\Sigma} \tag{31}$$

$$G = \frac{g}{E} = \beta_1\Sigma + \frac{\beta_2}{\beta_1}\Sigma_r \tag{32}$$

and

$$\alpha_1 = \frac{2(1-\nu)}{(1+\frac{2}{3}\mu)^2} \quad \alpha_2 = \frac{2\left[(1-2\nu) - \frac{1}{3}\mu(1+\nu)\right]}{\left(1+\frac{2}{3}\mu\right)^2} \quad \alpha_3 = \frac{1+\nu}{1+\frac{2}{3}\mu} \tag{33}$$

$$\beta_1 = \frac{1+\frac{2}{3}\eta}{1+\frac{2}{3}\mu} \qquad \beta_2 = \frac{\eta - \mu}{\left(1+\frac{2}{3}\mu\right)}\beta_1 \tag{34}$$

The integration of (30) is carried from infinity, where all stress components vanish, to any chosen value of the effective stress at the cavity $\Sigma(r = a) = \Sigma_a$. The corresponding value of the radial stress $\Sigma_r(r = a) = -p / E = -P$ determines the internal pressure associated with that particular value of $\Sigma_a$. Equation (30) is universal in the sense that each pair $(\Sigma_a, P)$ represents a state of stress at some radius within the medium. At infinity, where the field is fully elastic, it is permissible to replace (30) by its asymptotic form $\Sigma_r = -\frac{2}{3}\Sigma$. That relation provides convenient starting data for the numerical integration of (30).

Once the variation of $\Sigma_r$ with $\Sigma$ has been determined the circumferential stress follows from (15) as

$$\Sigma_\theta = \frac{\Sigma + \left(1 - \frac{1}{3}\mu\right)\Sigma_r}{\left(1 + \frac{2}{3}\mu\right)} \tag{35}$$

To obtain the spatial distribution of the stresses we combine (30) with (22), using (25) and (27), which gives

$$\frac{\frac{dr}{r}}{d\Sigma} = -\frac{1+\frac{2}{3}\mu}{2} \cdot \frac{\left(\alpha_1 + \beta_1 \frac{\Sigma}{G}\varepsilon_P' + \beta_2 \frac{\Sigma_r}{G^2}\varepsilon_P\right)\exp\left[\alpha_3(\Sigma - \mu\Sigma_r) + \frac{3}{2}\cdot\frac{\Sigma}{G}\varepsilon_P\right]}{\exp\left[\alpha_3(\Sigma - \mu\Sigma_r) + \frac{3}{2}\cdot\frac{\Sigma}{G}\varepsilon_P\right] - 1 + (\Sigma - \mu\Sigma_r)\left(\alpha_2 - \beta_2\frac{\Sigma}{G^2}\varepsilon_P\right)} \tag{36}$$

This equation when integrated from the cavity ($r=a$, $\Sigma=\Sigma_a$, $\Sigma_r=-P$) to any given radius results in the dependence of $\Sigma$ on r at any value of internal pressure. Transformation to the Eulerian radial coordinate $R=r+u$ is provided by (28) and the second of (21), viz

$$\ell n\frac{R}{r} = \frac{1+\frac{2}{3}\mu}{2}\left(\alpha_1\Sigma + \alpha_2\Sigma_r + \beta_1\frac{\Sigma}{G}\varepsilon_P\right) \tag{37}$$

To sum up, the solution of the cavity problem is reduced to the numerical integration of (30) and (36). The same pair of equations describes also the behaviour of a thick spherical shell under internal pressure with the outer surface being stress free.

## Some Special Cases

(a) Associated flow rule

When $\eta=\mu$ the plastic potential g coincides with the effective stress $\sigma_e$. The flow theory relations (17) then integrate exactly to the deformation theory relations (18) so no approximation is involved in the solution. From (34) we have that $\beta_1=1$ $\beta_2=0$ and the governing equation (30) becomes

$$\frac{d\Sigma_r}{d\Sigma} = -\frac{(\Sigma - \mu\Sigma_r)(\alpha_1 + \varepsilon_P')}{\exp\left[\alpha_3(\Sigma - \mu\Sigma_r) + \frac{3}{2}\varepsilon_P\right] - 1 + \alpha_2(\Sigma - \mu\Sigma_r)} \tag{38}$$

with a similar simplification of (36). For $\mu=0$ we recover from (38) the von-Mises quadrature solution (Durban and Baruch, 1976).

(b) Elastic/perfectly-plastic behaviour

While no definite yield point has been assumed so far in the analysis there is no difficulty in handling elasto/plastic response with discontinuous derivatives at the elastic/plastic boundary. At sufficiently low pressures the behaviour is elastic. When the pressure (termed $P_Y$) is attained, yielding begins at the cavity wall. At higher pressures the plastic zone spreads into the body. During the initial elastic phase $\varepsilon_P \equiv 0$ and (30) is reduced to

$$\frac{d\Sigma_r}{d\Sigma} = -\frac{\alpha_1(\Sigma - \mu\Sigma_r)}{\exp[\alpha_3(\Sigma - \mu\Sigma_r)] - 1 + \alpha_2(\Sigma - \mu\Sigma_r)} \tag{39}$$

This equation is of course completely independent of the plastic parameters and in fact can be recast into the quadrature form

$$\Sigma_r = -\int_0 \frac{2(1-\nu)(\Sigma_\theta - \Sigma_r)d(\Sigma_\theta - \Sigma_r)}{\exp[(1+\nu)(\Sigma_\theta - \Sigma_r)] - 1 + 2(1-2\nu)(\Sigma_\theta - \Sigma_r)} \tag{40}$$

where the remote stress-free state provides the lower limit on the integral. Expression (40) together with the elastic version of (36) -- which can likewise be reduced to a simple integral -- are the cavity expansion solution for the hypoelastic solid (7) as detailed in (Durban and Baruch (1974). If the elastic strains are sufficiently small it becomes possible to recover from (40) the linear elastic relation $\Sigma_r + 2\Sigma_\theta = 0$.

The onset of yielding is signalled by the equality $\Sigma = \Sigma_Y$ where $\Sigma_Y = Y / E$ with Y denoting the yield stress along the $\varepsilon_P - \sigma_e$ curve. Plastic flow will first occur at the cavity when the nondimensionalized pressure P reaches the value $P_Y$ as determined by the solution of (39). With further increase in pressure a plastic zone will spread radially from the cavity. Within the plastic zone we rewrite equation (30) with the effective strain as the independent variable (since $\Sigma \equiv \Sigma_Y$ and $d\Sigma \equiv 0$) as

$$\frac{d\Sigma_r}{d\varepsilon_P} = -\frac{(\Sigma_Y - \mu\Sigma_r)\beta_1 \frac{\Sigma_Y}{G}}{\exp\left[\alpha_3(\Sigma_Y - \mu\Sigma_r) + \frac{3}{2}\cdot\frac{\Sigma_Y}{G}\varepsilon_P\right] - 1 + (\Sigma_Y - \mu\Sigma_r)\left(\alpha_2 - \beta_2\frac{\Sigma_Y}{G^2}\varepsilon_P\right)} \tag{41}$$

where now

$$G = \beta_1\Sigma_Y + \frac{\beta_2}{\beta_1}\Sigma_r \tag{42}$$

The solution procedure starts with integrating (39) from infinity to the elastoplastic interface where $\Sigma = \Sigma_Y$ and $\Sigma_r = -Pi$ (the interface pressure). This value of $\Sigma_r$ along with $\varepsilon_P = 0$ are used as the initial data for integrating (41) from the interface inwards. The solution is again universal since to each cavity plastic strain $\varepsilon_{Pa}$ there corresponds a specific pressure P and each pair ($\varepsilon_{Pa}$, P) corresponds to a radius within the body. The spatial distribution of stresses and strains can be determined in a similar way from (36) with

$$\frac{\frac{dr}{r}}{d\Sigma} = -\frac{1+\frac{2}{3}\mu}{2} \cdot \frac{\alpha_1 \exp[\alpha_3(\Sigma - \mu\Sigma_r)]}{\exp[\alpha_3(\Sigma - \mu\Sigma_r)] - 1 + \alpha_2(\Sigma - \mu\Sigma_r)} \tag{43}$$

in the elastic zone, $0 \le \Sigma \le \Sigma_Y$. In the plastic zone $\Sigma = \Sigma_Y$ and

$$\frac{\frac{dr}{r}}{d\varepsilon_P} = -\frac{1+\frac{2}{3}\mu}{2} \cdot \frac{\beta_1 \frac{\Sigma_Y}{G} \exp\left[\alpha_3(\Sigma_Y - \mu\Sigma_r) + \frac{3}{2} \cdot \frac{\Sigma_Y}{G}\varepsilon_P\right]}{\exp\left[\alpha_3(\Sigma_Y - \mu\Sigma_r) + \frac{3}{2} \cdot \frac{\Sigma_Y}{G}\varepsilon_P\right] - 1 + (\Sigma_Y - \mu\Sigma_r)\left(\alpha_2 - \beta_2 \frac{\Sigma_Y}{G^2}\varepsilon_P\right)} \tag{44}$$

where G is given by (42). These differential equations are supplemented by (37) when transforming to the Eulerian spatial radial coordinate.

For the associated flow rule the elastic zone is still described by (39) and (43), but in the plastic zone where $\beta_1=1$, $\beta_2=0$ and $G \equiv \Sigma_Y$, eqs (41) and (44) are reduced to

$$\frac{d\Sigma_r}{d\varepsilon_P} = -\frac{\Sigma_Y - \mu\Sigma_r}{\exp\left[\alpha_3(\Sigma_Y - \mu\Sigma_r) + \frac{3}{2}\varepsilon_P\right] - 1 + \alpha_2(\Sigma_Y - \mu\Sigma_r)} \tag{45}$$

$$\frac{\frac{dr}{r}}{d\varepsilon_P} = -\frac{1+\frac{2}{3}\mu}{2} \cdot \frac{\exp\left[\alpha_3(\Sigma_Y - \mu\Sigma_r) + \frac{3}{2}\varepsilon_P\right]}{\exp\left[\alpha_3(\Sigma_Y - \mu\Sigma_r) + \frac{3}{2}\varepsilon_P\right] - 1 + \alpha_2(\Sigma_Y - \mu\Sigma_r)} \tag{46}$$

The stress field in the inner plastic domain has the same Eulerian spatial distribution for the non-associated flow and deformation theories by the following argument. Writing the equilibrium equation (20) as $d\sigma_r / dR + 2(\sigma_r - \sigma_\theta) / R = 0$ and combining this with (25) when $\sigma_e \equiv Y$ we find the profiles

$$\Sigma_r = \frac{\Sigma_Y}{\mu} - \left(P + \frac{\Sigma_Y}{\mu}\right)\left(\frac{A}{R}\right)^{\alpha_4} \qquad \Sigma_\theta = \frac{\Sigma_Y}{\mu} - \alpha_5\left(P + \frac{\Sigma_Y}{\mu}\right)\left(\frac{A}{R}\right)^{\alpha_4} \tag{47}$$

where A is the deformed radius of the cavity, and

$$\alpha_4 = \frac{2\mu}{1+\frac{2}{3}\mu} \qquad \alpha_5 = \frac{1-\frac{1}{3}\mu}{1+\frac{2}{3}\mu} \tag{48}$$

Furthermore, the plastic strain rates of the flow theory (17) form the identity

$$\dot{\varepsilon}_r^P + 2\beta_3\dot{\varepsilon}_\theta^P = 0 \qquad \text{with} \qquad \beta_3 = \frac{1-\frac{1}{3}\eta}{1+\frac{2}{3}\eta} \tag{49}$$

which integrates to the deformation theory (18) relation

$$\varepsilon_r^P + 2\beta_3\varepsilon_\theta^P = 0 \tag{50}$$

Upon superposing the elastic strains this is equivalent to

$$\varepsilon_r + 2\beta_3\varepsilon_\theta = (1-2\nu\beta_3)\Sigma_r - 2[\nu-(1-\nu)\beta_3]\Sigma_\theta \tag{51}$$

Recalling now that $\varepsilon_r = \ell n(dR/dr)$ and $\varepsilon_\theta = \ell n(R/r)$ and using the stresses (47) it becomes possible to integrate (51) and express the undeformed radius r as a quadrature of the deformed radius R,

$$r^{2\beta_3+1} - a^{2\beta_3+1} = (2\beta_3+1)\int_A^R \left\{ \exp\left[\beta_5\left(P+\frac{\Sigma_Y}{\mu}\right)\left(\frac{A}{R}\right)^{\alpha_4} - \beta_4\frac{\Sigma_Y}{\mu}\right]\right\} R^{2\beta_3}\, dR \tag{52}$$

where

$$\beta_4 = \frac{3(1-2\nu)}{1+\frac{2}{3}\eta} \qquad \beta_5 = \frac{3(1-2\nu)+\frac{2}{3}(1+\nu)\mu\eta}{\left(1+\frac{2}{3}\eta\right)\left(1+\frac{2}{3}\mu\right)} \tag{53}$$

The well-known small strain elastic solution for the outer elastic zone gives vanishing hydrostatic stress $\Sigma_r + 2\Sigma_\theta = 0$. Stress continuity at the elasto/plastic interface implies by (47) that

$$P+\frac{\Sigma_Y}{\mu} = \frac{1+\frac{2}{3}\mu}{\mu}\Sigma_Y\left(\frac{R_i}{A}\right)^{\alpha_4} \tag{54}$$

Also at the interface we have the circumferential strain

$$\ell n\frac{R_i}{r_i} = \left[-\nu\Sigma_r + (1-\nu)\Sigma_\theta\right]_{\Sigma=\Sigma_Y} \approx \frac{1+\nu}{3}\Sigma_Y \tag{55}$$

for small strain elasticity. Specifying (52) at the interface gives with the aid of (54), and the approximation $R_i \approx r_i$ from (55),

$$1-\left(\frac{a}{r_i}\right)^{2\beta_3+1} = \left[(2\beta_3+1)\exp\left(-\frac{\beta_4}{\mu}\Sigma_Y\right)\right]\int_{t_a}^{1}\left[\exp\left(\frac{1+\frac{2}{3}\mu}{\mu}\beta_5\Sigma_Y t^{-\alpha_4}\right)\right] t^{2\beta_3}dt \tag{56}$$

where $t = R/R_i$ and $t_a = A/R_i$.

Equation (56) provides the relation between $R_i/A$ and $r_i/a$ with the associated pressure given by (54). It is then a matter of ease to evaluate from (52) the relation between r and R for any given internal pressure. Thus, the stress and strain profiles (in both Eulerian and Lagrangian radial coordinates) are identical for the flow and deformation theories. The only difference between the two models lies in the radial distribution of the effective plastic strain $\varepsilon_P$.

(c) <u>The thin-shell thought experiment</u>

We may imagine the medium as being composed of adjacent infinite layers each behaving as a thin walled spherical shell under net outward pressure $\Delta P$. The differential equations (30) and (36) can be approximated by their finite differences analogues and combined to yield

$$\Delta P = \frac{2\left(\frac{h_o}{r_o}\right)}{1+\frac{2}{3}\mu}(\Sigma - \mu\Sigma_r)\exp\left[-\alpha_3(\Sigma - \mu\Sigma_r) - \frac{3}{2}\cdot\frac{\Sigma}{G}\varepsilon_P\right] \tag{57}$$

Here we have applied the approximation $dr / r \approx h_o / r_o$ where ($h_o$, $r_o$) are, respectively, the undeformed thickness and middle-surface radius of the shell, and it is understood that the r.h.s. of (57) represents the average value over the thickness of each layer. Now, for a hardening material, in the deep plastic range we may assume that $\Sigma >> \Sigma_r$ so that $G \approx \beta_1\Sigma$, and also neglect the elastic term from the exponent in (57). This leaves us with

$$\Delta P = \frac{2\left(\frac{h_o}{r_o}\right)}{1+\frac{2}{3}\mu}\Sigma\exp\left(-\frac{3\varepsilon_P}{2\beta_1}\right) \tag{58}$$

which has a maximum when

$$\Sigma\varepsilon_P' = \frac{2}{3}\beta_1 \tag{59}$$

For a power hardening law $\varepsilon_P = B\Sigma^n$, where (n,B) are material parameters, the effective stress and strain at the maximum pressure difference are

$$\Sigma = \left(\frac{2\beta_1}{3nB}\right)^{1/n} \qquad \varepsilon_P = \frac{2\beta_1}{3n} \tag{60}$$

with the corresponding pressure

$$\Delta P_{max} = \frac{2\left(\frac{h_o}{r_o}\right)}{1+\frac{2}{3}\mu}\left(\frac{2\beta_1}{3enB}\right)^{1/n} \tag{61}$$

Recalling that $\beta_1$ is given in (34) we find that the level of non-associativity has a considerable influence on the plastic response of the elemental spherical shell. Thus, the pair ($\mu$=0.5, $\eta$=1.5 $\Rightarrow$ $\beta_1$=1.5) produces an instability strain $\varepsilon_P$ which is three times higher than the one obtained with ($\mu$=1.5, $\eta$=0 $\Rightarrow$ $\beta_1$=0.5). The corresponding ratio of pressures (61) is $3^{1/n}$ which for a typical value of n=2 is ~1.73.

It is reasonable to expect that the sensitivity of the strength of a single shell to the ratio $\eta/\mu$ will be transmitted to the entire medium even though plastic collapse is not radially homogeneous.

(d) The Mohr-Coulomb solid

Existing studies on cavity expansion problems in soils are dominated by the elastic/perfectly-plastic Mohr-Coulomb model (Chadwick, 1959; Bigoni and Laudiero, 1990).

The Mohr-Coulomb solid -- in its non associated version -- is governed by the yield function and flow potential

$$\sigma_e = \bar{\mu}\sigma_1 - \sigma_3 \qquad g = \bar{\eta}\sigma_1 - \sigma_3 \tag{62}$$

where ($\sigma_1$, $\sigma_3$) are, respectively, the largest and smallest principal stresses, and ($\bar{\mu}$, $\bar{\eta}$) are material parameters. For the spherical cavity problem ($\sigma_1 = \sigma_\theta = \sigma_\varphi, \sigma_3 = \sigma_r$) and the plastic strain rates become

$$\dot{\varepsilon}_r^P = -\frac{\sigma_e}{g}\dot{\varepsilon}_\theta^P \qquad \dot{\varepsilon}_\theta^P = \frac{1}{2}\bar{\eta}\frac{\sigma_e}{g}\dot{\varepsilon}_P$$

with

$$\sigma_e = \bar{\mu}\sigma_\theta - \sigma_r \qquad g = \bar{\eta}\sigma_\theta - \sigma_r \tag{63}$$

These relations are on an equal footing with the Drucker-Prager constitutive law (15)-(17) apart from a rescaling of the material parameters. It follows that the preceding analysis can be adapted to include the non-associated Mohr-Coulomb solid with strain hardening.

**Spontaneous Growth**

The internal pressure approaches an asymptotic limit $P_c$ which, when attained, will induce a spontaneous expansion of the cavity at a constant load $P=P_c$. That cavitation pressure can be obtained from the loading path (30) at the limit of $\Sigma_a \to \infty$. For elastic/perfectly-plastic solids we integrate the pair (39) and (41) with $\varepsilon_{Pa} \to \infty$. We can use also (56) directly with $a / r_i \to 0$ to find the cavitation limit of $t_a = A/R_i$ from

$$\int_{t_a}^{1}\left[\exp\left(\frac{1+\frac{2}{3}\mu}{\mu}\beta_5\Sigma_Y t^{-\alpha_4}\right)\right]t^{2\beta_3}dt = \frac{\exp\left(\frac{\beta_4}{\mu}\Sigma_Y\right)}{2\beta_3+1} \tag{64}$$

The cavitation pressure follows from (54) as

$$P_c = \frac{\Sigma_Y}{\mu}\left[\left(1+\frac{2}{3}\mu\right)t_a^{-\alpha_4}-1\right] \tag{65}$$

for both flow and deformation theories.

The self similarity of the spontaneous growth state enables a direct evaluation of the cavitation pressure with no need to trace the entire straining history (Fleck et al., 1992). The procedure is particularly helpful with the flow theory (17) where an elaborated step by step incremental routine is required for the complete numerical solution.

All field quantities are assumed to depend on the single coordinate $\xi$=R/A, with the time like differentiation

$$(\dot{\ }) = \left(\frac{\dot{R}}{A}-\xi\frac{\dot{A}}{A}\right)\frac{d(\ )}{d\xi} \tag{66}$$

The strain rates are now $\dot{\varepsilon}_r = d\dot{R}/dR$ and $\dot{\varepsilon}_\theta = \dot{R}/R$ and the constitutive relations (17) take the form

$$\frac{dV}{d\xi} = (V-\xi)\left[\frac{d}{d\xi}(\Sigma_r - 2\nu\Sigma_\theta) + \left(-1+\frac{1}{3}\eta\right)\frac{\Sigma}{G}\cdot\frac{d\varepsilon_P}{d\xi}\right] \tag{67a}$$

$$\frac{V}{\xi} = (V-\xi)\left[\frac{d}{d\xi}(-\nu\Sigma_r + (1-\nu)\Sigma_\theta) + \left(\frac{1}{2}+\frac{1}{3}\eta\right)\frac{\Sigma}{G}\cdot\frac{d\varepsilon_P}{d\xi}\right] \tag{67b}$$

where $V = \dot{R}/\dot{A}$.

The couple (67) together with the equilibrium requirement

$$\frac{d\Sigma_r}{d\xi} = \frac{2}{\xi}(\Sigma_\theta - \Sigma_r) \tag{68}$$

provide three equations for $\Sigma_r$, $\Sigma_\theta$, V as unknowns. Integration can start at the remote linear elastic field where

$$\xi \to \infty: \quad \Sigma_r = -\frac{C}{\xi^3} \quad \Sigma_\theta \to \frac{C}{2\xi^3} \quad V \to \frac{3(1+\nu)C}{2\xi^2} \tag{69}$$

and C is an unknown constant. At the cavity boundary we require

$$\xi = 1: \quad V = 1 \quad \Sigma_\theta \to \infty \quad \Sigma_r = -P_c\,. \tag{70}$$

A standard shooting method can be used to solve this two point boundary value problem.

Alternatively, we may exploit the coordinate transformation

$$X = \frac{V}{\xi} \tag{71}$$

and eliminate $\xi$ between (67b) and (68) to obtain

$$\Sigma_r' = \frac{(X-1)(\Sigma - \mu\Sigma_r)\left(\alpha_1 + \beta_1 \frac{\Sigma}{G}\varepsilon_P'\right)}{X - \alpha_2(X-1)(\Sigma - \mu\Sigma_r)} \tag{72}$$

Similarly, equation (67a) is transformed to

$$X' = (X-1)\left[\alpha_3(\mu\Sigma_r' - 1) - \frac{3}{2}\cdot\frac{\Sigma}{G}\varepsilon_P'\right] \tag{73}$$

where $(\,)' = d(\,)/d\Sigma$.

The system (72)-(73) is simpler to integrate numerically in comparison with (67)-(69). In the far field $X = (1+\nu)\Sigma$, $\Sigma_r = -\frac{2}{3}\Sigma$, and integration can start with any reasonable choice of $\Sigma$ in the remote elastic zone. At the cavity $\Sigma \to \infty$ $X \to 1$ and $\Sigma_r = -P_c$.

For the associated flow rule ($\Sigma$=G) equation (73) integrates exactly to

$$\ell n(1-X) = -\alpha_3(\Sigma - \mu\Sigma_r) - \frac{3}{2}\varepsilon_P \tag{74}$$

Inserting this relation in (72) gives exactly equation (38) with the cavity data $\Sigma \to \infty$ at $X=1$. This shows formally that the cavitation limit of the loading history is the self-similar field of spontaneous growth. A similar analysis can be made for the non-associated deformation theory (with hardening), and for the non-associated flow theory with the elastic/perfectly-plastic response. For the latter case we deduce from (49) and (67)

$$\frac{dV}{d\xi}+2\beta_3\frac{V}{\xi}=(V-\xi)\alpha_4\beta_5\left(P+\frac{\Sigma_Y}{\mu}\right)\xi^{-\alpha_4-1} \tag{75}$$

The solution of this equation together with the cavity and interface conditions leads again to (64).

The similarity solution can be extended to include inertia effects: Radial motion is governed by the equation ($\rho$ being the density)

$$\frac{d\sigma_r}{dR}+\frac{2}{R}(\sigma_r-\sigma_\theta)=\rho\ddot{R} \tag{76}$$

which can be rewritten in the nondimensionalized form

$$\frac{d\Sigma_r}{d\xi}+\frac{2}{\xi}(\Sigma_r-\Sigma_\theta)=\frac{\rho\dot{A}^2}{E}(V-\xi)\frac{dV}{d\xi} \tag{77}$$

Conservation of matter requires that $\dot{\rho}+\rho\,\underset{\sim}{I}\cdot\cdot\underset{\sim}{D}=0$ or, in a rate form,

$$\frac{\dot{\rho}}{\rho}+\dot{\varepsilon}_r+2\dot{\varepsilon}_\theta=0 \tag{78}$$

Hence, in steady growth,

$$(V-\xi)\frac{d\ell n\frac{\rho}{\rho_o}}{d\xi}+\frac{dV}{d\xi}+2\frac{V}{\xi}=0 \tag{79}$$

where $\rho_o$ is material density in the undeformed state. The governing system consists now of (77), (79) and (67) or their analogue for the deformation theory.

In the absence of hardening we can replace (67) in the plastic zone by

$$\frac{dV}{d\xi}+2\beta_3\frac{V}{\xi}=(V-\xi)\left[(1-2\nu\beta_3)\frac{d\Sigma_r}{d\xi}-2(\nu-(1-\nu)\beta_3)\frac{d\Sigma_\theta}{d\xi}\right] \tag{80}$$

along with the yield condition

$$\left(1+\frac{2}{3}\mu\right)\Sigma_\theta-\left(1-\frac{1}{3}\mu\right)\Sigma_r=\Sigma_Y \tag{81}$$

Relations (80)-(81) are valid also with the deformation theory and it follows that both theories will predict identical stresses and strain rates (but different $\dot{\varepsilon}_P$) in dynamic spontaneous growth when the material is elastic/perfectly-plastic. A full analysis of dynamic spontaneous growth goes beyond the scope of this paper.

## Numerical Examples

Equation (30) has been integrated numerically for the power law material $\varepsilon_P = B\Sigma^n$ with B=100 and n=2 (which are representative for soils). Typical pressure-expansion curves are displayed in Fig. 1 and it is clearly seen that a cavitation limit is approached with increasing cavity strain A/a. The cavitation pressure is higher for larger values of $\mu$ and $\eta$. Similar conclusions hold for the elastic/perfectly-plastic model (Fig. 2). The curves in Fig. 2 were evaluated from (39) and (41) with a yield stress of $\Sigma_Y = Y/E = 0.01$. These ideally-plastic results are valid, as we have already explained, for both flow and deformation theories.

The variation of the cavitation pressure $P_c$ with the level of non-associativity $\eta/\mu$ is depicted in Fig. 3 for the power law material, and in Fig. 4 for the elastic/perfectly-plastic model. In the latter case flow and deformation theories give identical values for the cavitation pressure. With the hardening model (Fig. 3) the differences in $P_c$ predicted by the two theories are practically negligible. Thus, the limiting pressure obtained from the deformation theory loading path (30) is virtually the same as the one given by the flow theory spontaneous growth equations (72)-(73). It can be concluded from Figs. 3-4 that the cavitation pressure increases with $\eta$, but the dependence on $\mu$ is more complex with a possible inversion for negative values of $\eta$.

The upshot of these sample calculations appears to be a considerable sensitivity of material response to the relative pressure dependence of the yield surface and the flow potential. For the associated case ($\eta=\mu$), in particular, there is a remarkable strengthening of the medium with increasing $\mu$. The ratio of the cavitation pressures for $\mu$=1.5 and $\mu$ =0.5 is over 2.5 for n=2, and over 4.5 for the elastic/perfectly-plastic material.

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## Figure Captions

Fig. 1: Pressure-expansion curves for a power-law material. $\varepsilon_P = B\Sigma^n$ with B = 100 and n = 2. $\nu = 0.3$. Results are for the deformation theory.

Fig. 2: Pressure-expansion curves for an elastic/perfectly-plastic material. $\Sigma_Y = 0.01$. $\nu = 0.3$. Results are for both flow and deformation theories.

Fig. 3: Variation of the cavitation pressure with level of non-associativity for a power-law material. $\varepsilon_P = B\Sigma^n$ with B = 100 and n = 2. $\nu = 0.3$. Results for deformation and flow theories are practically the same.

Fig. 4: Variation of the cavitation pressure with level of non-associativity for an elastic/perfectly-plastic material. $\Sigma_Y = 0.01$. $\nu = 0.3$. Results are for both flow and deformation theories.

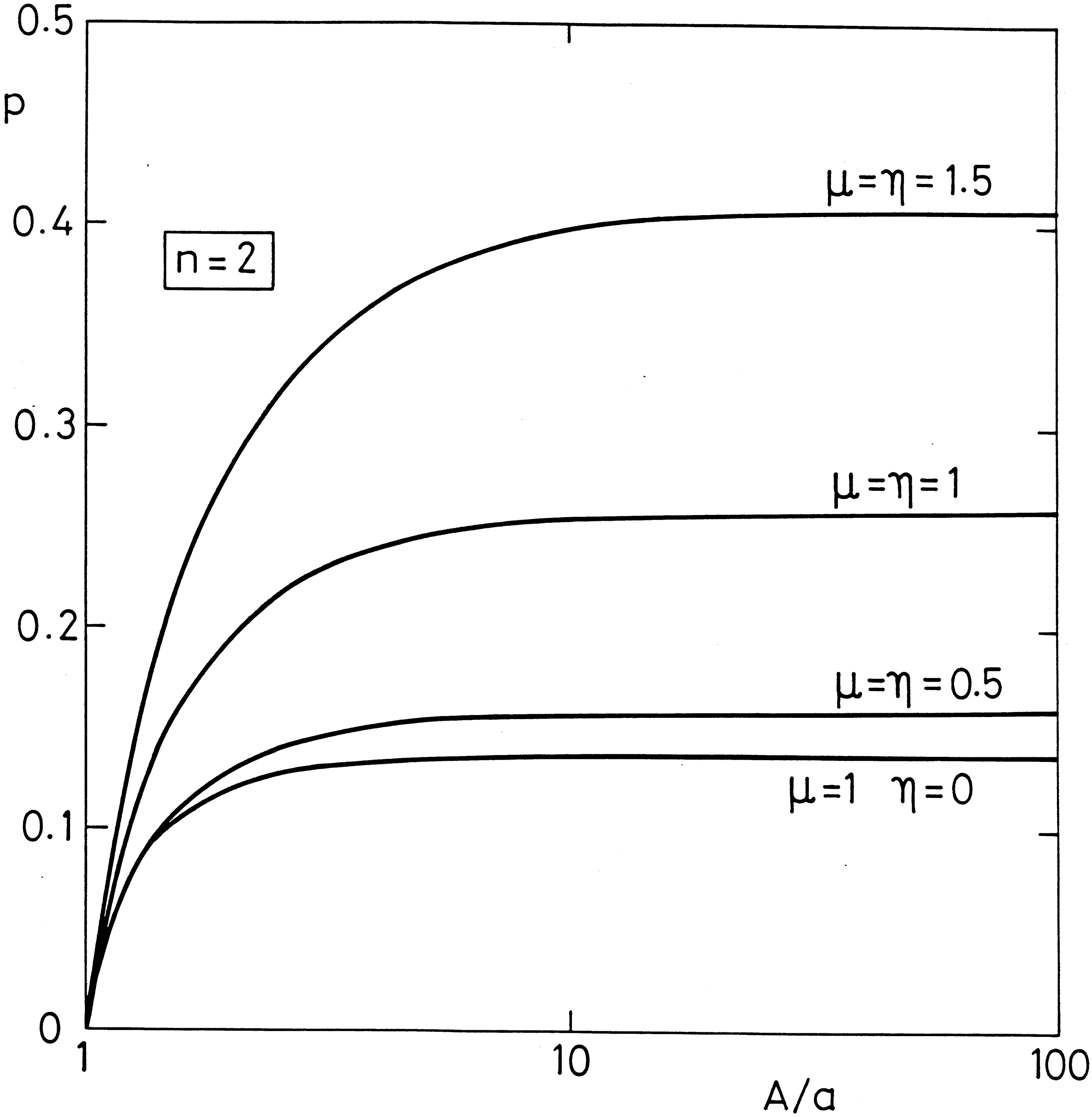


Fig. 1: Pressure-expansion curves for a power-law material. $\varepsilon_p = B\Sigma^n$ with B = 100 and n = 2. ν = 0.3. Results are for the deformation theory.

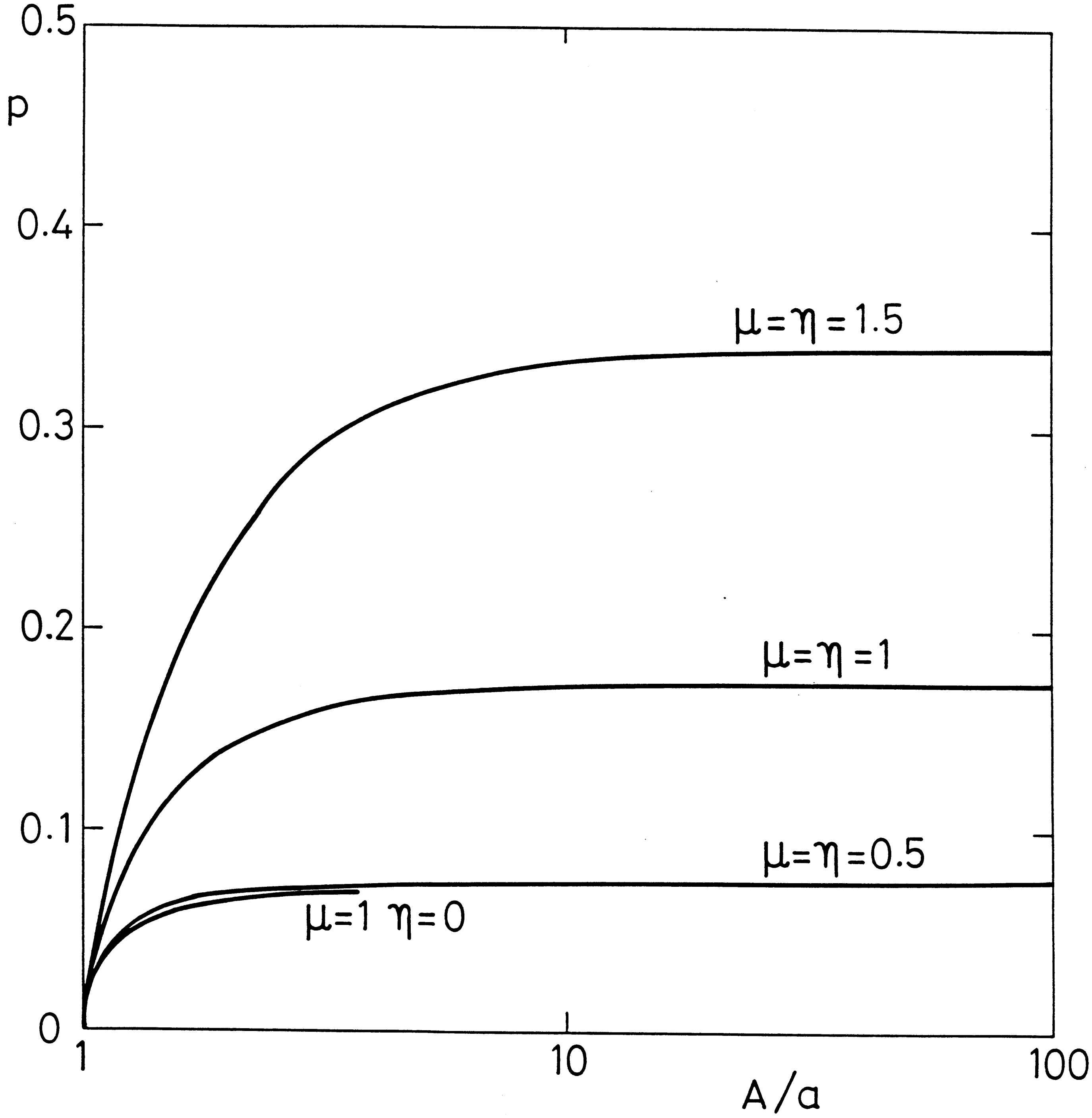


Fig. 2: Pressure-expansion curves for an elastic/perfectly-plastic material. $\Sigma_Y = 0.01$. $\nu = 0.3$. Results are for both flow and deformation theories.

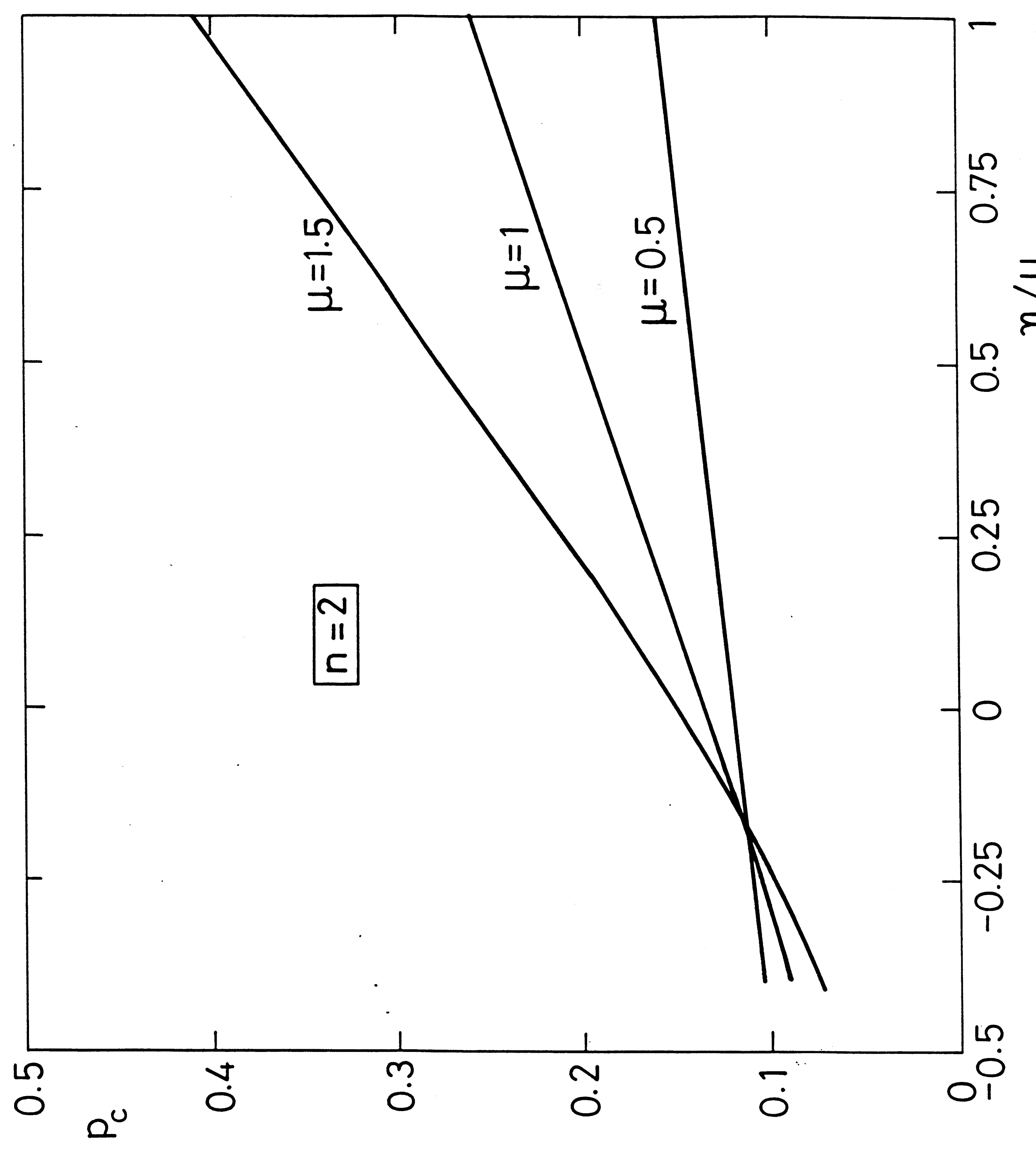


Fig. 3: Variation of the cavitation pressure with level of non-associativity for a power-law material. $\varepsilon_P = B\Sigma^n$ with B = 100 and n = 2. $\nu = 0.3$. Results for deformation and flow theories are practically the same.

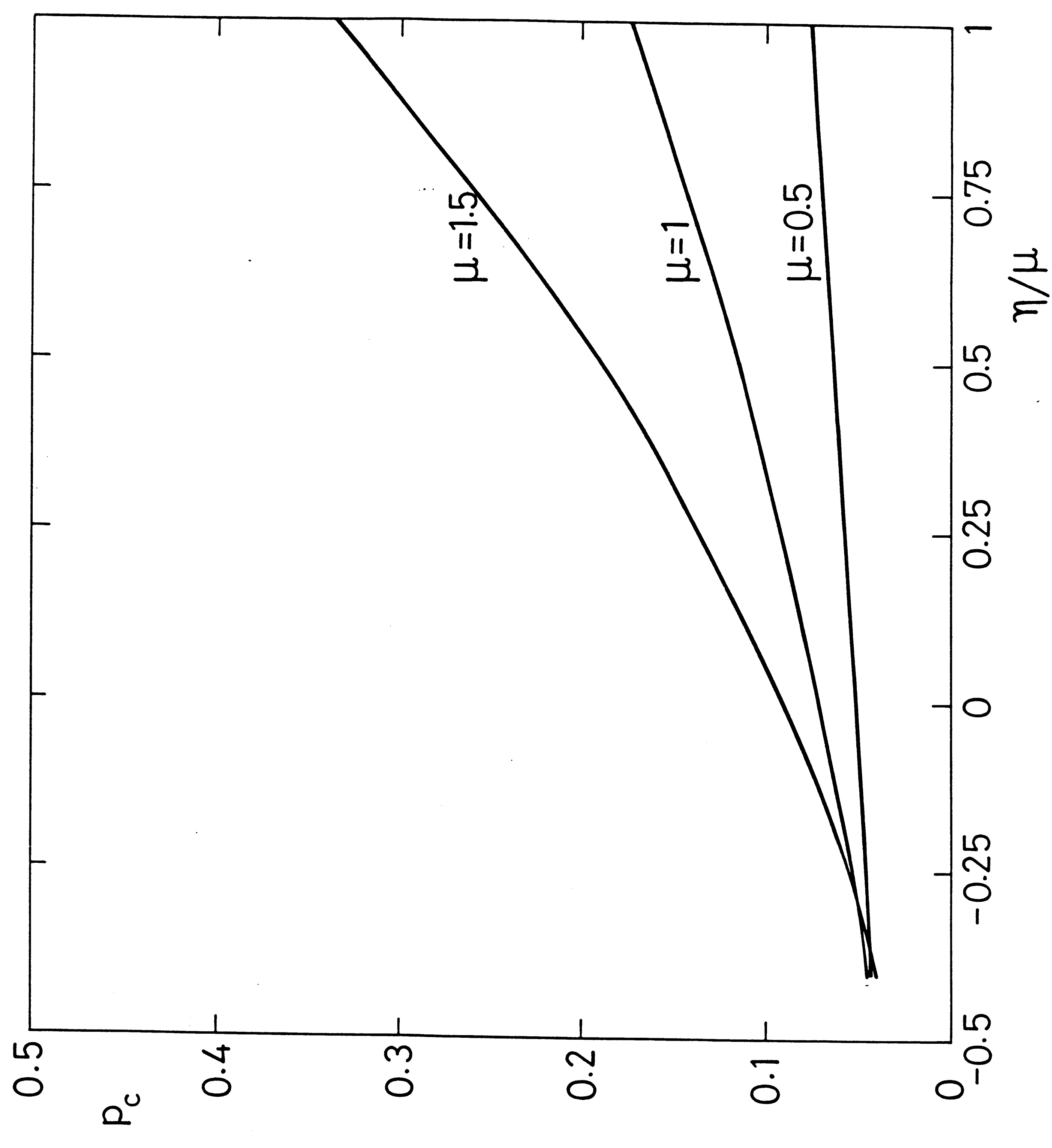


Fig. 4: Variation of the cavitation pressure with level of non-associativity for an elastic/perfectly-plastic material. $\Sigma_Y = 0.01$. $\nu = 0.3$. Results are for both flow and deformation theories.