Pergamon

J. Mech. Phys. Solids, Vol. 42, No. 12, pp. 1851–1882, 1994

0022–5096/94 $7.00+0.00

0022–5096(94)00057–3

BOUNDS AND ESTIMATES FOR LINEAR COMPOSITES WITH STRAIN GRADIENT EFFECTS

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(*Received* 18 *April* 1994; *in revised form* 15 *August* 1994)

ABSTRACT

Overall mechanical properties are studied for linear composites demonstrating a size effect. Variational principles of Hashin–Shtrikman type are formulated for incompressible composites involving the gradient of strain in their constitutive description. These variational principles are applied to linear, statistically homogeneous and isotropic two-phase composites. Upper and lower bounds of Hashin–Shtrikman type for the effective shear modulus and related self-consistent estimates are derived in terms of volume fraction and a two-point correlation function accounting for the scale of microstructure. An alternative self-consistent scheme for matrix-inclusion strain-gradient composites is also proposed by a development of the approach laid down by Budiansky and Hill. Some numerical results are given to demonstrate the size effect.

1. INTRODUCTION

Several observed elastic and plastic phenomena display a size effect whereby the smaller the size the stronger the response. A discussion of the experimental evidence and relevant literature can be found e.g. in Fleck and Hutchinson (1993) and Fleck *et al.* (1994).

Significant progress has been made over the last three decades in the development of homogenization theories for the prediction of the macroscopic behaviour of composites from their microstructure. Conventionally, these macroscopic theories are based on constitutive descriptions which operate with dimensionless strain and therefore give predictions which are independent of the scale of the microstructure. For example, the strength of particle reinforced composites is predicted to be independent of particle size, and the strength of fine-grained polycrystalline metals is predicted to be the same as that of coarse-grained metals.

To take the scale effect into account one usually includes gradients of strain in the constitutive description (Koiter, 1964; Mindlin, 1965; Aifantis, 1984, 1987, 1992; Muhlhaus and Aifantis, 1991; Zbib and Aifantis, 1992; Fleck and Hutchinson, 1993; Fleck *et al.*, 1994). This results in the introduction of a material length scale l into the constitutive relations for dimensional consistency.

Fleck and Hutchinson (1993) have suggested a mathematically convenient strain gradient constitutive model. They assumed that the strain energy function (for both linear and nonlinear materials) is dependent on the local strain and curvature in a simple way. The resulting constitutive relations reduce to the conventional form in the limit $l \to 0$. Fleck and Hutchinson also estimated the strengthening due to the introduction of a dilute random distribution of equi-sized rigid inclusions of radius a, and the softening due to a dilute concentration of spherical voids. The size effect was found to be particularly strong for the rigid inclusions.

In the present paper we focus on homogenization for two-phase linear "strain-gradient" composites where the concentrations are not dilute. Starting from the variational principles introduced first by Koiter (1964) and then generalized in a straightforward way by Fleck and Hutchinson (1993), we formulate variational principles of Hashin and Shtrikman (1962) type by introducing a linear strain gradient comparison medium, following the general strategy of Willis (1977, 1983, 1991). Explicit analysis of Green's function and of related operators in the strain gradient context is crucial for the derivation of bounds of Hashin–Shtrikman type; these bounds give the overall response of statistically homogeneous and isotropic linear composites subjected to uniform macroscopic strain. The bounds contain the constitutive parameters of the phases, volume fractions *and also* the integrals involving the correlation functions which account for the size effect. Explicit expressions for the bounds are given for a simple choice of correlation function. Following Willis (1977), self-consistent estimates for the effective shear modulus are derived using the proposed Hashin–Shtrikman procedure.

Alternative self-consistent estimates are derived for "particulate" composites by assuming that one phase constitutes the matrix and the other is a random distribution of spherical inclusions; the approach is a development of that laid down by Budiansky (1965) and Hill (1965). The theoretical developments in this paper for "strain-gradient" composites share some common formal features with those in time-harmonic dynamics discussed by Sabina and Willis (1988).

The structure of the paper is as follows. First, the constitutive formulation is given and minimum principles are restated from Fleck and Hutchinson (1993). Variational principles of Hashin–Shtrikman type are derived from these minimum principles. Next, Hashin–Shtrikman bounds are obtained for linear statistically homogeneous and isotropic two-phase composites. Self-consistent estimates are obtained from both the Hashin–Shtrikman procedure and the "spherical inclusion" assumption. Numerical results are reported for the stiffness of two-phase composites, and some general conclusions are drawn.

2. CONSTITUTIVE LAW, MINIMUM PRINCIPLES AND ELEMENTARY BOUNDS

Although the purpose of this paper is to study *linear* incompressible strain gradient composites, we shall formulate the basic ideas in a more general nonlinear form; this requires little extra effort and has the virtue of application to nonlinear composites.

Let $\mathbf{u}$ be a displacement field and $\varepsilon_{ij} \equiv (u_{i,j}+u_{j,i})/2$ be the related infinitesimal strain†, in terms of cartesian coordinates x_i. Incompressibility implies that $\boldsymbol{\varepsilon}$ is devi-

† A comma in the subscript denotes differentiation, and a repeated suffix denotes summation from 1 to 3; e_{ijk} is the alternating tensor.

atoric, i.e. $\varepsilon_{ii} = 0$. The unsymmetric deviatoric curvature tensor is related to the rotation vector $\theta_i \equiv \frac{1}{2}e_{ijk}u_{k,j}$ via

$$\chi_{ij} \equiv \theta_{i,j} = e_{ikl}\varepsilon_{jl,k} = \tfrac{1}{2}e_{ikl}u_{l,kj}, \tag{2.1}$$

and can be decomposed into its symmetric and antisymmetric parts:

$$\chi_{ij} = \chi^{\mathrm{s}}_{ij} + \chi^{\mathrm{a}}_{ij} \tag{2.2}$$

with

$$\chi^{\mathrm{s}}_{ij} = \tfrac{1}{2}(\chi_{ij} + \chi_{ji}), \quad \chi^{\mathrm{a}}_{ij} = \tfrac{1}{2}(\chi_{ij} - \chi_{ji}).$$

In a strain gradient deformation theory for incompressible media the strain energy function w is assumed to be a strictly convex function of both the (deviatoric) strain tensor $\boldsymbol{\varepsilon}$ and the curvature tensor $\boldsymbol{\chi}$ [related to the strain gradient via (2.1)]:

$$w = w(\boldsymbol{\varepsilon}, \boldsymbol{\chi}). \tag{2.3}$$

A dependence of w on the curvature $\boldsymbol{\chi}$ has the effect of introducing a material length scale l into the constitutive law for dimensional consistency, which will be illustrated later. To simplify technical details we will assume henceforth that only the *symmetric* part of $\boldsymbol{\chi}$ contributes to (2.3). In the remainder of the paper the superscripts s are omitted to keep the notation as simple as possible. The symmetric curvature is therefore introduced as

$$\chi_{ij} = \tfrac{1}{2}(\theta_{i,j} + \theta_{j,i}) = \tfrac{1}{4}(e_{ikl}u_{l,kj} + e_{jkl}u_{l,ki}). \tag{2.4}$$

The form (2.4) has the advantage that it gives rise to macroscopic effective composite properties of the same functional form: the composite possesses a single effective length scale l_*. The general form (2.1) suffers from the drawback that it gives rise to effective composite properties with *two* values for the composite length scale l_{s} and $l_{\mathrm{a}}\,(\neq l_{\mathrm{s}})$ corresponding to contributions from symmetric and antisymmetric components respectively.

Deviatoric stress $\mathbf{s}$ and couple stress $\mathbf{m}$ are defined as work conjugates of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\chi}$:

$$s_{ij} = \frac{\partial w}{\partial \varepsilon_{ij}} \quad m_{ji} = \frac{\partial w}{\partial \chi_{ij}}. \tag{2.5}$$

Note that the assumption (2.4) implies that the couple stress $\mathbf{m}$ is symmetric.

A stress potential $\phi(\boldsymbol{\sigma}, \mathbf{m})$ can be introduced as the dual of $w(\boldsymbol{\varepsilon}, \boldsymbol{\chi})$:

$$\phi(\boldsymbol{\sigma}, \mathbf{m}) = \max_{\boldsymbol{\varepsilon}, \boldsymbol{\chi}} \{\sigma_{ij}\varepsilon_{ij} + m_{ji}\chi_{ij} - w(\boldsymbol{\varepsilon}, \boldsymbol{\chi})\}. \tag{2.6}$$

Here $\boldsymbol{\sigma}$ is the *symmetric* stress related to its deviatoric part $\mathbf{s}$ via

$$\sigma_{ij} = s_{ij} + \delta_{ij}\sigma_{\mathrm{h}} \tag{2.7}$$

and $\sigma_{\mathrm{h}}(\mathbf{x})$ is the hydrostatic stress. In (2.6) the maximum is taken over all possible $\boldsymbol{\varepsilon}$ and $\boldsymbol{\chi}$; ϕ is finite and unique since w is assumed to be strictly convex in $\boldsymbol{\varepsilon}$ and $\boldsymbol{\chi}$.

As an example, in a constitutive law suggested by Fleck and Hutchinson (1993) w depends on $\boldsymbol{\varepsilon}$ and $\boldsymbol{\chi}$ via a scalar measure $\mathscr{E}$:

$$\mathscr{E}^2 = \varepsilon_e^2 + l^2 \chi_e^2. \tag{2.8}$$

Here, $\varepsilon_e \equiv \sqrt{\frac{2}{3}\varepsilon_{ij}\varepsilon_{ij}}$ is the von Mises strain invariant. The measure $\chi_e \equiv \sqrt{\frac{2}{3}\chi_{ij}^s\chi_{ij}^s}$ is the analogous curvature invariant; the length scale l is required on dimensional grounds.

With the choice $w = w(\mathscr{E})$, the stress potential $\phi(\boldsymbol{\sigma}, \mathbf{m})$ assumes the form

$$\phi = \phi(\Sigma) = \max_{\mathscr{E}} \{\Sigma\mathscr{E} - w(\mathscr{E})\},$$

where the scalar stress measure Σ is defined as the work conjugate to $\mathscr{E}$:

$$\Sigma = \frac{\mathrm{d}w}{\mathrm{d}\mathscr{E}} = (\sigma_e^2 + l^{-2} m_e^2)^{1/2}. \tag{2.9}$$

In (2.9) $\sigma_e \equiv \sqrt{\frac{3}{2}s_{ij}s_{ij}}$ is the usual von Mises effective stress and $m_e \equiv \sqrt{\frac{3}{2}m_{ij}m_{ij}}$ is the analogous effective couple stress. In the limit $l \to 0$ the given constitutive description reduces to conventional J_2 deformation theory.

2.1. *Minimum principles*

Consider now a composite material, i.e. a microscopically heterogeneous but macroscopically homogeneous solid, occupying a volume V, subjected to prescribed displacements and rotations

$$\mathbf{u} = \hat{\mathbf{u}}; \quad \boldsymbol{\theta} = \hat{\boldsymbol{\theta}} \tag{2.10}$$

over the boundary S. It is assumed that $w(\mathbf{x}; \boldsymbol{\varepsilon}, \boldsymbol{\chi})$ may vary through the volume. It is natural also to introduce a kinematic assumption of continuity of the displacement $\mathbf{u}$ and the rotation $\boldsymbol{\theta}$ throughout V, including at a finite number of interfaces S_{int}. In the case of a two-phase composite this is equivalent to the statement that the phases are perfectly bonded.

Postulating the stationary principle for the energy integral

$$W(\mathbf{u}) = \int_V w(\mathbf{x}; \boldsymbol{\varepsilon}, \boldsymbol{\chi})\, \mathrm{d}\mathbf{x} \tag{2.11}$$

for any kinematically admissible incompressible displacement field $\mathbf{u}$ [subject to (2.10)] provides

$$\delta W(\mathbf{u}) = \int_V \{\sigma_{ji}\delta u_{i,j} + m_{ij}\delta\theta_{i,j}\}\, \mathrm{d}\mathbf{x} = -\int_V \{\sigma_{ji,j} + \tau_{ji,j}\}\delta u_i\, \mathrm{d}\mathbf{x}$$
$$+ \int_{S_{\text{int}}} [m_{ji}n_j\delta\theta_i + \{\sigma_{ij}n_j + \tau_{ji}n_j\}\delta u_i]\, \mathrm{d}s(\mathbf{x}) = 0. \tag{2.12}$$

The vector $\mathbf{n}$ is the unit normal to the interface, and $[\cdot]$ denotes a jump of the bracketed function across the interface S_{int}.

We have introduced within (2.12) the tensor τ_{ij}, defined by

$$\tau_{jk} \equiv -\tfrac{1}{2}e_{ijk}m_{pi,p}. \tag{2.13}$$

In couple stress theory τ_{ij} is the antisymmetric part of the stress tensor, and (2.13) is a restatement of moment equilibrium.

It follows immediately from (2.12) that within the volume V the equation (of equilibrium)

$$\sigma_{ji,j} + \tau_{ji,j} = 0 \tag{2.14}$$

must be satisfied; the stress traction $(\sigma_{ji} + \tau_{ji})\, n_j$ and the couple stress traction $m_{ji}n_j$ must also be continuous across any interface.

The stationary principle (2.12) gives a *minimum* principle provided $w(\boldsymbol{\varepsilon}, \boldsymbol{\chi})$ is strictly convex with respect to $\boldsymbol{\varepsilon}$ and $\boldsymbol{\chi}$ [see Fleck and Hutchinson (1993) for further details]. Denote the displacement field which makes $W(\mathbf{u})$ stationary by $\mathbf{u}^*$, and denote the associated value of W by $\tilde{W}$. Then,

$$\tilde{W} \equiv W(\mathbf{u}^*) < W(\mathbf{u}) \tag{2.15}$$

for any kinematically admissible $\mathbf{u}$ which differs from $\mathbf{u}^*$.

To formulate the *complementary* (linear or nonlinear) minimum principle, define the complementary energy by

$$\Phi(\boldsymbol{\sigma}, \mathbf{m}) = \int_V \phi(\boldsymbol{\sigma}, \mathbf{m})\, \mathrm{d}\mathbf{x} \tag{2.16}$$

with ϕ given by (2.6). Consider a composite of volume V with

$$(\sigma_{ij} + \tau_{ij})\, n_i = T_j^0; \quad m_{ij}n_i = q_j^0$$

for prescribed tractions $(\mathbf{T}^0, \mathbf{q}^0)$ on the boundary S. Then Φ has a minimum value $\tilde{\Phi} \equiv \Phi(\boldsymbol{\sigma}^*, \mathbf{m}^*)$ for the "true" field $(\boldsymbol{\sigma}^*, \mathbf{m}^*)$. For all other statically admissible fields $(\boldsymbol{\sigma}, \mathbf{m})$ we have

$$\tilde{\Phi} \leqslant \Phi(\boldsymbol{\sigma}, \mathbf{m}) \tag{2.17}$$

provided ϕ is strictly convex in $\boldsymbol{\sigma}$ and $\mathbf{m}$; the relation (2.17) holds equality only if $\boldsymbol{\sigma} = \boldsymbol{\sigma}^*$, $\mathbf{m} = \mathbf{m}^*$.

Examples of material behaviour which satisfy the convexity conditions are the power law materials:

$$w(\boldsymbol{\varepsilon}, \boldsymbol{\chi}) = \frac{n}{n+1} \Sigma_0 \mathscr{E}_0 \left(\frac{\mathscr{E}}{\mathscr{E}_0}\right)^{(n+1)/n}, \tag{2.18}$$

where Σ_0 and $\mathscr{E}_0$ are normalizing constants and $n \geqslant 1$ is the hardening index;† $\mathscr{E}$ is the scalar measure (2.3).

For *linear* strain gradient solids the energy density assumes the form

$$w(\boldsymbol{\varepsilon}, \boldsymbol{\chi}) = \mu \varepsilon_{ij}\varepsilon_{ij} + \mu l^2 \chi_{ij}\chi_{ij}, \tag{2.19}$$

† This corresponds to the power-law relation $\mathscr{E}/\mathscr{E}_0 = (\Sigma/\Sigma_0)^n$ between the strain and the stress measures.

where μ is the shear modulus. This gives the following simple form for the linear constitutive relations:

$$s_{ij} = 2\mu\,\varepsilon_{ij} \quad m_{ij} = 2\mu l^2 \chi_{ij}. \tag{2.20}$$

2.2. *Elementary bounds*

The minimum principles (2.15) and (2.17) provide elementary bounds for the averaged response of a two-phase composite in terms of volume fractions c_1 and c_2 as follows. Assume that phase number one of the composite has material properties (μ_1, l_1) and phase number two has properties (μ_2, l_2).

Consider the two-phase composite of overall volume V, subjected to the "uniform strain" boundary conditions

$$u_i^0(\mathbf{x}) = \varepsilon_{ij}^0 x_j, \quad \boldsymbol{\theta}_0(\mathbf{x}) = \mathbf{0} \tag{2.21}$$

on S. [If V were filled with a homogeneous medium then $\mathbf{u}(\mathbf{x}) = \mathbf{u}_0(\mathbf{x})$ would be the "true" field within V and corresponds to a uniform strain ε_{ij}^0.] Assuming that the composite is linear and isotropic, the effective shear modulus μ_* can be defined via

$$\mu_* \varepsilon_{ij}^0 \varepsilon_{ij}^0 = \frac{1}{|V|} W(\mathbf{u}^*),$$

where $\mathbf{u}^*$ is the actual field in the composite associated with the prescribed displacement boundary conditions. The Voigt bound for μ_* follows by substituting $\mathbf{u}_0(\mathbf{x})$ into the right side of (2.15):

$$\mu_* \leqslant c_1\mu_1 + c_2\mu_2. \tag{2.22}$$

In analogous fashion, now prescribe "uniform stress" traction boundary conditions $T_j^0 = \sigma_{ij}^0 n_j, \mathbf{q}^0 = 0$ on S. Then, the *complementary* variational principle (2.17) gives the Reuss lower bound

$$\mu_* \geqslant (c_1\mu_1^{-1} + c_2\mu_2^{-1})^{-1}. \tag{2.23}$$

We note that the elementary results (2.22) and (2.23) are not influenced by couple-stress effects and provide *uniform* bounds for μ_* with respect to the length parameters l_1, l_2. We emphasize that the macroscopic strain $\boldsymbol{\varepsilon}^0$ is taken to be constant and therefore the macroscopic curvature vanishes. The *microscopic* curvature, however, does not vanish due to local inhomogeneities, which gives rise to the scale effect at the macroscale.

Improved bounds displaying the scale effect can be found from the variational principles of Hashin–Shtrikman (1962) type which are derived for strain-gradient composites in the next section. We shall derive them in a form which does not necessarily assume that the phases are linear. Thus, the variational statements may also be useful for deriving bounds for *nonlinear* composites (cf. Willis, 1983, 1991).

3. VARIATIONAL PRINCIPLES OF HASHIN–SHTRIKMAN TYPE

We shall follow a general strategy developed by Willis (1977, 1983, 1991) for both linear and nonlinear composites. First, introduce a homogeneous *linear comparison* *medium* with material properties μ_0 and l_0 and described by the strain energy function $w_0(\boldsymbol{\varepsilon}, \boldsymbol{\chi})$ given by (2.19).

3.1. *Minimum principle for strain energy of composite*

If w_0 is chosen in such a way that at each point of the composite $(w_0 - w)$ grows faster than linearly when $\boldsymbol{\varepsilon}$ or $\boldsymbol{\chi}$ is large, then a function $U(\boldsymbol{\pi}, \boldsymbol{\beta})$ can be derived as the dual to $(w - w_0)$,

$$U(\boldsymbol{\pi}, \boldsymbol{\beta}; \mathbf{x}) = \min_{\boldsymbol{\varepsilon}, \boldsymbol{\chi}} \{\varepsilon_{ij}\pi_{ij} + \chi_{ij}\beta_{ij} - w(\boldsymbol{\varepsilon}, \boldsymbol{\chi}; \mathbf{x}) + w_0(\boldsymbol{\varepsilon}, \boldsymbol{\chi})\}. \tag{3.1}$$

In (3.1) we have introduced arbitrary symmetric "polarizations" π_{ij}, β_{ij}, whose physical interpretation will be made clear later.

It follows immediately from (3.1) that†

$$w(\boldsymbol{\varepsilon}, \boldsymbol{\chi}; \mathbf{x}) \leqslant w_0(\boldsymbol{\varepsilon}, \boldsymbol{\chi}) + \boldsymbol{\varepsilon} \cdot \boldsymbol{\pi} + \boldsymbol{\chi} \cdot \boldsymbol{\beta} - U(\boldsymbol{\pi}, \boldsymbol{\beta}; \mathbf{x}). \tag{3.2}$$

Recall that the minimum principle (2.15) states that the actual strain energy $\tilde{W}$ of the composite of volume V satisfies

$$\tilde{W} \leqslant \int_V \mathrm{w}(\boldsymbol{\varepsilon}, \boldsymbol{\chi}; \mathbf{x}) \, \mathrm{d}\mathbf{x}$$

for all kinematically admissible fields $\boldsymbol{\varepsilon}$ and $\boldsymbol{\chi}$, as long as $w(\boldsymbol{\varepsilon}, \boldsymbol{\chi}; \mathbf{x})$ is strictly convex for every $\mathbf{x}$. Making use of (3.2) this inequality can be re-expressed in the form,

$$\tilde{W} \leqslant \int_V \{w_0(\boldsymbol{\varepsilon}, \boldsymbol{\chi}) + \boldsymbol{\varepsilon} \cdot \boldsymbol{\pi} + \boldsymbol{\chi} \cdot \boldsymbol{\beta} - U(\boldsymbol{\pi}, \boldsymbol{\beta}; \mathbf{x})\} \, \mathrm{d}\mathbf{x} \tag{3.3}$$

for arbitrary $\boldsymbol{\pi}$, $\boldsymbol{\beta}$ and for all kinematically admissible $\boldsymbol{\varepsilon}$ and $\boldsymbol{\chi}$. We consider (3.3) to be the starting point from which is derived the Hashin–Shtrikman upper bound.

When $(w_0 - w)$ is strictly convex and smooth, the equality holds in (3.3) if and only if $(\boldsymbol{\varepsilon}, \boldsymbol{\chi})$ is the actual field and

$$\begin{aligned} \pi_{ij} &= s_{ij} - 2\mu_0 \varepsilon_{ij}, \\ \beta_{ij} &= m_{ij} - 2\mu_0 l_0^2 \chi_{ij} \end{aligned} \tag{3.4}$$

for the true fields $\mathbf{s}$ and $\mathbf{m}$; this establishes the connection with the standard definition of polarizations. The relations (3.4) appear by extremizing the right hand side of (3.1) with respect to $(\boldsymbol{\varepsilon}, \boldsymbol{\chi})$ and using definitions (2.5) of $\mathbf{s}$ and $\mathbf{m}$.

In the further analysis of (3.3) we follow Willis (1977, 1983). First, we select the displacement field $\mathbf{u}$ which makes stationary the right-hand side of (3.3), for any given $\boldsymbol{\pi}$ and $\boldsymbol{\beta}$. This leads to an upper bound for $\tilde{W}$ in terms of (arbitrary) polarizations $\boldsymbol{\pi}$ and $\boldsymbol{\beta}$ only. In the next section we optimize (3.3) further with respect to the polarizations.

On writing the right hand side of (3.3) as $I(\boldsymbol{\pi}, \boldsymbol{\beta}; \mathbf{u})$, we find the stationary value of I by taking its variation with respect to incompressible fields $\mathbf{u}$ and keeping $(\boldsymbol{\pi}, \boldsymbol{\beta})$ fixed:

† Henceforth the scalar product $\mathbf{f} \cdot \mathbf{g}$ means $f_{ij}g_{ij}$ for relevant $\mathbf{f}$ and $\mathbf{g}$.

$$\delta I(\boldsymbol{\pi}, \boldsymbol{\beta}; \mathbf{u}) = \int_V \{(2\mu_0 \varepsilon_{ij} + \pi_{ij})\delta\varepsilon_{ij} + (2\mu_0 l_0^2 \chi_{ij} + \beta_{ij})\delta\chi_{ij}\}\, \mathrm{d}\mathbf{x} = 0, \tag{3.5}$$

where $\delta\varepsilon_{ij} = \frac{1}{2}(\delta u_{i,j} + \delta u_{j,i})$ and $\delta\chi_{ij} = \frac{1}{4}(e_{ikl}\delta u_{l,kj} + e_{jkl}\delta u_{l,ki})$. This gives after integration by parts

$$\int_V ([L_0(\nabla)\mathbf{u}]_i + \pi_{ji,j} - \tfrac{1}{2} e_{pji}\beta_{pq,qj})\,\delta u_i\, \mathrm{d}\mathbf{x} = 0, \tag{3.6}$$

where

$$[L_0(\nabla)\mathbf{u}]_i \equiv \mu_0 u_{i,pp} - \tfrac{1}{4}\mu_0 l_0^2 e_{psi} e_{prj} u_{j,qqrs}. \tag{3.7}$$

Further rearrangement of (3.7) for incompressible fields $\mathbf{u}$ provides

$$L_0(\nabla)\mathbf{u} = \mu_0 \Delta\mathbf{u} - \tfrac{1}{4}\mu_0 l_0^2 \Delta^2 \mathbf{u}, \tag{3.8}$$

where Δ is the Laplace operator.

Since (3.6) is satisfied for an arbitrary *incompressible* field $\delta\mathbf{u}$ (i.e. $\delta u_{i,i} = 0$), it implies that

$$[L_0(\nabla)\mathbf{u}]_i + \sigma_{\mathrm{h},i} + \pi_{ji,j} - \tfrac{1}{2} e_{pji}\beta_{pq,qj} = 0, \tag{3.9}$$

where σ_{h} is the hydrostatic stress.

The differential equation (3.9) must be supplemented by the boundary conditions (2.10) in order to solve it for $\mathbf{u}$ and σ_{h}.

We solve the linear equation (3.9) by assuming that the displacement field $\mathbf{u}$ is the superposition of the solution to two problems, written as $\mathbf{u} = \mathbf{u}_0 + \tilde{\mathbf{u}}$.

(i) The comparison medium is subjected to the displacement boundary conditions (2.10); the displacement field $\mathbf{u}_0$ satisfies

$$[L_0(\nabla)\mathbf{u}_0]_i + \sigma^0_{\mathrm{h},i} = 0, \tag{3.10}$$

where $\sigma^0_{\mathrm{h},i}$ is the hydrostatic stress field.

(ii) The comparison medium is subjected to a distribution of body forces f_i of magnitude

$$f_i \equiv \pi_{ji,j} - \tfrac{1}{2} e_{pji}\beta_{pq,qj} \tag{3.11}$$

with vanishing displacement and rotation on the boundary. The displacement field within the body $\tilde{\mathbf{u}}$ obeys the governing relation

$$[L_0(\nabla)\tilde{\mathbf{u}}]_i + \tilde{\sigma}^0_{\mathrm{h},i} + f_i = 0, \tag{3.12}$$

where $\tilde{\sigma}_{\mathrm{h}}$ is the associated hydrostatic stress.

The fields $\boldsymbol{\varepsilon}$ and $\boldsymbol{\chi}$ associated with the solution $\mathbf{u} = \mathbf{u}_0 + \tilde{\mathbf{u}}$ are

$$\begin{aligned} \boldsymbol{\varepsilon} &= \boldsymbol{\varepsilon}_0 - S\boldsymbol{\pi} - M\boldsymbol{\beta}, \\ \boldsymbol{\chi} &= \boldsymbol{\chi}_0 - P\boldsymbol{\pi} - Q\boldsymbol{\beta}, \end{aligned} \tag{3.13}$$

where S, M, P and Q are linear operators to be found. These operators are defined such that for arbitrary $\boldsymbol{\pi}$ and $\boldsymbol{\beta}$, $\tilde{\boldsymbol{\varepsilon}} = -S\boldsymbol{\pi} - M\boldsymbol{\beta}$ and $\tilde{\boldsymbol{\chi}} = -P\boldsymbol{\pi} - Q\boldsymbol{\beta}$ are the strain and the curvature derived from the displacement field $\tilde{\mathbf{u}}$, for the comparison body subjected to vanishing displacements and rotations on the boundary, and loaded by the body force (3.11). Appropriate approximate forms for the linear operators S, M, P and Q are given in Section 4.

To proceed, we substitute the solution (3.13) for $(\boldsymbol{\varepsilon}, \boldsymbol{\chi})$ into the right hand side of (3.3). It is shown in Appendix A that the first term on the right hand side becomes

$$\int_V w_0(\boldsymbol{\varepsilon}, \boldsymbol{\chi}) = \tilde{W}_0 + \tfrac{1}{2}\Gamma(\boldsymbol{\pi}, \boldsymbol{\beta}), \tag{3.14}$$

where

$$\Gamma(\boldsymbol{\pi}, \boldsymbol{\beta}) \equiv \langle \boldsymbol{\pi}, S\boldsymbol{\pi} \rangle + \langle \boldsymbol{\pi}, M\boldsymbol{\beta} \rangle + \langle \boldsymbol{\beta}, P\boldsymbol{\pi} \rangle + \langle \boldsymbol{\beta}, Q\boldsymbol{\beta} \rangle \tag{3.15}$$

and $\tilde{W}_0$ is the strain energy of the field $\mathbf{u}_0$ for the case where the volume V is filled with the comparison medium. Here the notation $\langle \cdot, \cdot \rangle$ denotes an inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_V f_{ij}(\mathbf{x}) g_{ij}(\mathbf{x})\, \mathrm{d}\mathbf{x} = \int_V \mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x})\, \mathrm{d}\mathbf{x} \tag{3.16}$$

for any relevant $\mathbf{f}$ and $\mathbf{g}$.

In summary, relation (3.3) can be formulated as the following minimum principle of Hashin–Shtrikman type for the strain energy $\tilde{W}$:

$$\tilde{W} \leqslant \tilde{W}_0 - \tfrac{1}{2}\Gamma(\boldsymbol{\pi}, \boldsymbol{\beta}) + \langle \boldsymbol{\varepsilon}_0, \boldsymbol{\pi} \rangle + \langle \boldsymbol{\chi}_0, \boldsymbol{\beta} \rangle - \int_V U(\boldsymbol{\pi}, \boldsymbol{\beta}; \mathbf{x})\, \mathrm{d}\mathbf{x} \tag{3.17}$$

for arbitrary polarization fields $(\boldsymbol{\pi}, \boldsymbol{\beta})$. For $(w - w_0)$ strictly convex the equality sign holds if and only if $\boldsymbol{\pi}$ and $\boldsymbol{\beta}$ are related to the "true" field via (3.4).

So far, we have assumed the comparison medium is linear but have made no such assumption for the composite. For a linear composite (3.17) simplifies since $U(\boldsymbol{\pi}, \boldsymbol{\beta}; \mathbf{x})$ can then be written in an explicit form. On writing $w(\boldsymbol{\varepsilon}, \boldsymbol{\chi})$ for the linear composite as

$$w(\boldsymbol{\varepsilon}, \boldsymbol{\chi}) = \mu(\mathbf{x})\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} + \mu(\mathbf{x}) l(\mathbf{x})^2 \boldsymbol{\chi} \cdot \boldsymbol{\chi},$$

at each point $\mathbf{x}$ of the volume, then the dual function $U(\boldsymbol{\pi}, \boldsymbol{\beta}; \mathbf{x})$ as defined by (3.1) is finite provided

$$\mu < \mu_0, \quad \mu l^2 < \mu_0 l_0^2, \tag{3.18}$$

and is expressed as

$$U(\boldsymbol{\pi}, \boldsymbol{\beta}; \mathbf{x}) = \frac{1}{4(\mu - \mu_0)} \boldsymbol{\pi} \cdot \boldsymbol{\pi} + \frac{1}{4(\mu l^2 - \mu_0 l_0^2)} \boldsymbol{\beta} \cdot \boldsymbol{\beta}. \tag{3.19}$$

Even in the more general case of nonlinear composites, further optimization of (3.17) is possible with respect to piecewise constant polarizations $\boldsymbol{\pi}$ and $\boldsymbol{\beta}$. Following the pattern of Willis (1983, 1991) these may eventually lead to nonlinear Hashin–Shtrikman bounds. This line of enquiry requires a separate investigation and is not pursued further here.

We note, however, that the upper bound (3.17) is simplified dramatically if a two-phase *linear* composite is considered to be statistically homogeneous and isotropic (see Section 4).

3.2. *Maximum principle for strain energy of composite*

Lower bounds for $\tilde{W}$ are obtained by choosing a comparison medium such that $(w-w_0)$ grows faster than linearly at large $(\boldsymbol{\varepsilon}, \boldsymbol{\chi})$ and by introducing the dual function U_- as

$$U_-(\boldsymbol{\pi}, \boldsymbol{\beta}; \mathbf{x}) = \max_{\boldsymbol{\varepsilon}, \boldsymbol{\chi}} \{\boldsymbol{\varepsilon}\cdot\boldsymbol{\pi}+\boldsymbol{\chi}\cdot\boldsymbol{\beta}-w(\boldsymbol{\varepsilon}, \boldsymbol{\chi}; \mathbf{x})+w_0(\boldsymbol{\varepsilon}, \boldsymbol{\chi})\}$$

[compare to (3.1)]. This yields

$$W(\mathbf{u}) \equiv \int_V w(\boldsymbol{\varepsilon}, \boldsymbol{\chi}; \mathbf{x})\,\mathrm{d}\mathbf{x} \geqslant \int_V \{w_0(\boldsymbol{\varepsilon}, \boldsymbol{\chi})+\boldsymbol{\varepsilon}\cdot\boldsymbol{\pi}+\boldsymbol{\chi}\cdot\boldsymbol{\beta}-U_-(\boldsymbol{\pi}, \boldsymbol{\beta}; \mathbf{x})\}\,\mathrm{d}\mathbf{x}.$$

The minimum principle (2.15) provides

$$\tilde{W} = \inf_{\mathbf{u}} W(\mathbf{u}) \geqslant \inf_{\mathbf{u}} \int_V \{w_0(\boldsymbol{\varepsilon}, \boldsymbol{\chi})+\boldsymbol{\varepsilon}\cdot\boldsymbol{\pi}+\boldsymbol{\chi}\cdot\boldsymbol{\beta}-U_-(\boldsymbol{\pi}, \boldsymbol{\beta}; \mathbf{x})\}\,\mathrm{d}\mathbf{x} \tag{3.20}$$

with $\mathbf{u}$ varying over all kinematically admissible states. Applying again the above optimization procedure to the right hand side of (3.20) we arrive at a lower bound for $\tilde{W}$ given by

$$\tilde{W} \geqslant \tilde{W}_0 - \tfrac{1}{2}\Gamma(\boldsymbol{\pi}, \boldsymbol{\beta}) + \langle \boldsymbol{\varepsilon}_0, \boldsymbol{\pi}\rangle + \langle \boldsymbol{\chi}_0, \boldsymbol{\beta}\rangle - \int_V U_-(\boldsymbol{\pi}, \boldsymbol{\beta}; \mathbf{x})\,\mathrm{d}\mathbf{x}, \tag{3.21}$$

which is valid for arbitrary fields $\boldsymbol{\pi}, \boldsymbol{\beta}$.

For the case of a *linear* composite U_- is determined by the formula (3.19) where, however, μ_0 and l_0 must satisfy

$$\mu > \mu_0, \quad \mu l^2 > \mu_0 l_0^2. \tag{3.22}$$

Similar methods can be developed for bounding of the *complementary* energy Φ starting from the complementary minimum principle (2.17); this is not pursued here.

4. LINEAR ISOTROPIC HASHIN–SHTRIKMAN BOUNDS

Specialize to the linear case. Then (3.19) holds and the relations (3.17) for the upper bound on $\tilde{W}$ and (3.21) for the lower bound can be transformed via (3.15) to

$$\begin{aligned} 2\tilde{W}_0 - 2\tilde{W} \underset{(\leqslant)}{\geqslant} \int_V \{&\tfrac{1}{2}(\mu-\mu_0)^{-1}\boldsymbol{\pi}(\mathbf{x})\cdot\boldsymbol{\pi}(\mathbf{x}) + \tfrac{1}{2}(\mu l^2 - \mu_0 l_0^2)^{-1}\,\boldsymbol{\beta}(\mathbf{x})\cdot\boldsymbol{\beta}(\mathbf{x}) \\ &+ \boldsymbol{\pi}(\mathbf{x})\cdot(S\boldsymbol{\pi})(\mathbf{x}) + \boldsymbol{\beta}(\mathbf{x})\cdot(Q\boldsymbol{\beta})(\mathbf{x}) + \boldsymbol{\pi}(\mathbf{x})\cdot(M\boldsymbol{\beta})(\mathbf{x}) \\ &+ \boldsymbol{\beta}(\mathbf{x})\cdot(P\boldsymbol{\pi})(\mathbf{x}) - 2\boldsymbol{\pi}(\mathbf{x})\cdot\boldsymbol{\varepsilon}_0(\mathbf{x}) - 2\boldsymbol{\beta}(\mathbf{x})\cdot\boldsymbol{\chi}_0(\mathbf{x})\}\,\mathrm{d}\mathbf{x}. \end{aligned} \tag{4.1}$$

The $\geqslant$ sign relates to the upper bound, and the $\leqslant$ is for the lower bound. Both inequalities become *equalities* if and only if $\boldsymbol{\pi}$ and $\boldsymbol{\beta}$ are the "true" polarizations (3.4).

So far, no statistical homogeneity has been employed, and the variational statement (4.1) is valid for arbitrary microstructure. Then, following Willis (1977, 1981) translation-invariant approximations for the non-local operators S, M, P and Q emerge, provided that the composite is statistically uniform and that the polarizations $\boldsymbol{\pi}$ and $\boldsymbol{\beta}$ oscillate rapidly about their mean values $\bar{\boldsymbol{\pi}}$ and $\bar{\boldsymbol{\beta}}$. By a standard development of the reasoning given by Willis (1977, 1981) it can be shown (see Appendix A) that for large V and for $\mathbf{x}$ remote from the boundary of the composite,

$$
\begin{aligned}
(S\boldsymbol{\pi})(\mathbf{x}) &\cong \int_V S^{\infty}(\mathbf{x}'-\mathbf{x})(\boldsymbol{\pi}(\mathbf{x}')-\bar{\boldsymbol{\pi}})\,\mathrm{d}\mathbf{x}',\\
(M\boldsymbol{\beta})(\mathbf{x}) &\cong \int_V M^{\infty}(\mathbf{x}'-\mathbf{x})(\boldsymbol{\beta}(\mathbf{x}')-\bar{\boldsymbol{\beta}})\,\mathrm{d}\mathbf{x}',\\
(P\boldsymbol{\pi})(\mathbf{x}) &\cong \int_V P^{\infty}(\mathbf{x}'-\mathbf{x})(\boldsymbol{\pi}(\mathbf{x}')-\bar{\boldsymbol{\pi}})\,\mathrm{d}\mathbf{x}',\\
(Q\boldsymbol{\beta})(\mathbf{x}) &\cong \int_V Q^{\infty}(\mathbf{x}'-\mathbf{x})(\boldsymbol{\beta}(\mathbf{x}')-\bar{\boldsymbol{\beta}})\,\mathrm{d}\mathbf{x}',
\end{aligned}
\tag{4.2}
$$

where the kernels S^{∞}, M^{∞}, P^{∞} and Q^{∞} are given by

$$
\begin{aligned}
S^{\infty}_{ijkl}(\mathbf{x}) &= -G_{il,kj}(\mathbf{x})_{|(ij),(kl)}, \quad M^{\infty}_{ijkl}(\mathbf{x}) = \tfrac{1}{4}e_{kqp}G_{ip,lqj}(\mathbf{x})_{|(ij),(kl)},\\
P^{\infty}_{ijkl}(\mathbf{x}) &= -\tfrac{1}{4}e_{irs}G_{sl,kjr}(\mathbf{x})_{|(ij),(kl)}, \quad Q^{\infty}_{ijkl}(\mathbf{x}) = \tfrac{1}{16}e_{irs}e_{kqp}G_{sp,lqjr}(\mathbf{x})_{|(ij),(kl)}.
\end{aligned}
\tag{4.3}
$$

Here, $G(\mathbf{x})$ is the infinite body Green's function for the comparison medium with parameters μ_0, l_0. An explicit expression for $G(\mathbf{x})$ is derived in Appendix B. In (4.3) and henceforth, (ij) denotes symmetrization with respect to the relevant indices.

Choose the polarizations $\boldsymbol{\pi}$ and $\boldsymbol{\beta}$ to be constant within each phase, i.e.

$$
\begin{aligned}
\boldsymbol{\pi}(\mathbf{x}) &= \boldsymbol{\pi}_1 f_1(\mathbf{x}) + \boldsymbol{\pi}_2 f_2(\mathbf{x}),\\
\boldsymbol{\beta}(\mathbf{x}) &= \boldsymbol{\beta}_1 f_1(\mathbf{x}) + \boldsymbol{\beta}_2 f_2(\mathbf{x}).
\end{aligned}
\tag{4.4}
$$

Here $f_j(\mathbf{x})$, $j = 1, 2$, are characteristic functions for the composite: $f_1(\mathbf{x}) = 1$ if $\mathbf{x}$ belongs to phase one; $f_1(\mathbf{x}) = 0$ otherwise; and $f_2(\mathbf{x}) = 1 - f_1(\mathbf{x})$.

By substitution of approximations (4.2) into (4.1) the terms containing the operators S, M, P and Q are simplified provided the composite is statistically uniform and isotropic. We avoid here a detailed discussion and refer the reader to the review of Willis (1981). A typical term in (4.1) transforms as follows

$$
\begin{aligned}
I \equiv \int_V \boldsymbol{\pi}(\mathbf{x})\cdot(S\boldsymbol{\pi})(\mathbf{x})\,\mathrm{d}\mathbf{x} &\simeq \sum_{r,s=1}^{2} \boldsymbol{\pi}_r \left\{\int_V f_r(\mathbf{x}) \int_{V^{\infty}} S^{\infty}(\mathbf{x}'-\mathbf{x})[f_s(\mathbf{x}')-c_s]\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x}\right\}\boldsymbol{\pi}_s\\
&= \sum_{r,s=1}^{2} \boldsymbol{\pi}_r \left\{\int_V f_r(\mathbf{x}) \int_{V^{\infty}} S^{\infty}(\mathbf{z})[f_s(\mathbf{x}+\mathbf{z})-c_s]\,\mathrm{d}\mathbf{z}\,\mathrm{d}\mathbf{x}\right\}\boldsymbol{\pi}_s,
\end{aligned}
$$

where V^{∞} is the infinite volume.

The cross-correlation function $\Psi_{rs}(\mathbf{z})$ describes the statistical distribution of phases, and is defined by

$$\Psi_{rs} \equiv \frac{1}{|V|}\int_V f_r(\mathbf{x})f_s(\mathbf{x}+\mathbf{z})\,\mathrm{d}\mathbf{x}.$$

The assumption of *statistical isotropy* implies that Ψ_{rs} depends only on the modulus of $\mathbf{z}$. The volume fraction of each phase can be introduced as

$$c_r = \Psi_{rr}(0) = \int_V f_r(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

As a result,

$$I \simeq \sum_{r,s=1}^{2} \boldsymbol{\pi}_r \left\{\int_{V^\infty} S^\infty(\mathbf{z})[\Psi_{rs}(\mathbf{z}) - c_r c_s]\,\mathrm{d}\mathbf{z}\right\}\boldsymbol{\pi}_s.$$

The correlations $\Psi_{rs}(|\mathbf{z}|)$ satisfy at each point $\mathbf{z}$ the conditions

$$\Psi_{11}+\Psi_{12} = \Psi_{11}+\Psi_{21} = c_1, \quad \Psi_{21}+\Psi_{22} = c_2, \quad c_1+c_2 = 1,$$

and are therefore uniquely defined in terms of a "correlation coefficient" $h(|\mathbf{z}|)$ such that $h(0) = 1$, and

$$\Psi_{11} = c_1^2 + c_1c_2\,\mathrm{h}; \quad \Psi_{12} = \Psi_{21} = c_1c_2(1-h); \quad \Psi_{22} = c_2^2 + c_1c_2h$$

(see e.g. Willis, 1985). It is also required that $h(|\mathbf{z}|) \to 0$ when $|\mathbf{z}| \to \infty$ to ensure absence of long range order.

It is observed from (4.3) and the properties of $G(\mathbf{x})$ (Appendix B) that M and P are odd functions of $\mathbf{x}$; they give no contribution to (4.1) for constant polarizations and isotropic correlations. Finally, (4.1) transforms into the algebraic relation

$$\begin{aligned} 2\tilde{W}_0 - 2\tilde{W} \underset{(\leqslant)}{\geqslant} |V|\Bigg[&\sum_{r=1}^{2} \tfrac{1}{2}c_r(\mu_r-\mu_0)^{-1}\boldsymbol{\pi}_r\cdot\boldsymbol{\pi}_r \\ &+ \sum_{r,s=1}^{2}(\delta_{rs}-c_r)\,c_s\boldsymbol{\pi}_r\cdot\hat{S}\boldsymbol{\pi}_s - 2\sum_{r=1}^{2} c_r\boldsymbol{\pi}_r\cdot\bar{\boldsymbol{\varepsilon}}_0\Bigg] \\ +|V|\Bigg[\sum_{r=1}^{2}\tfrac{1}{2}c_r(\mu_r l_r^2 - \mu_0 l_0^2)^{-1}\boldsymbol{\beta}_r\cdot\boldsymbol{\beta}_r &+ \sum_{r,s=1}^{2}(\boldsymbol{\delta}_{rs}-c_r)\,c_s\boldsymbol{\beta}_r\cdot\hat{Q}\boldsymbol{\beta}_s - 2\sum_{r=1}^{2} c_r\boldsymbol{\beta}_r\cdot\bar{\boldsymbol{\chi}}_0\Bigg]. \end{aligned} \quad (4.5)$$

The upper inequality holds so long as (3.18) is satisfied; the lower inequality holds provided the condition (3.22) is met. Here

$$\hat{S} \equiv \int_{V^\infty} S^\infty(\mathbf{z})h(|\mathbf{z}|)\,\mathrm{d}\mathbf{z}, \quad \hat{Q} \equiv \int_{V^\infty} Q^\infty(\mathbf{z})\,h(|\mathbf{z}|)\,\mathrm{d}\mathbf{z} \quad (4.6)$$

are algebraic four-tensors, and $\bar{\boldsymbol{\varepsilon}}_0$ and $\bar{\boldsymbol{\chi}}_0$ are mean values, given by

$$\bar{\boldsymbol{\varepsilon}}_0 \equiv \frac{1}{|V|}\int_V \boldsymbol{\varepsilon}_0(\mathbf{x})\,\mathrm{d}\mathbf{x}, \quad \bar{\boldsymbol{\chi}}_0 \equiv \frac{1}{|V|}\int_V \boldsymbol{\chi}_0(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

In (4.5)

$$\tilde{W}_0 = \mu_0\langle \boldsymbol{\varepsilon}_0, \boldsymbol{\varepsilon}_0\rangle + \mu_0 l_0^2 \langle \boldsymbol{\chi}_0, \boldsymbol{\chi}_0\rangle, \tag{4.7}$$

where the inner product has already been defined by (3.16). In order to extract upper and lower bounds on the effective shear modulus μ_*, take $\boldsymbol{\varepsilon}_0 =$ constant and $\boldsymbol{\chi}_0 = \mathbf{0}$. Optimization of the bounds (4.5) with respect to choice of the piecewise constant polarizations $\boldsymbol{\pi}_r$, $\boldsymbol{\beta}_r$ then follows the same formal algebraic routine as that employed by Willis (1977). The result is

$$\tilde{W} \underset{(\geqslant)}{\leqslant} \bar{\mu}\, \boldsymbol{\varepsilon}_0 \cdot \boldsymbol{\varepsilon}_0, \tag{4.8}$$

where

$$\bar{\mu} = \left[\sum_{r=1}^{2} c_r \{1+4(\mu_r-\mu_0)\mu_S\}^{-1}\right]^{-1} \sum_{j=1}^{2} c_j \mu_j [1+4(\mu_j-\mu_0)\mu_S]^{-1}. \tag{4.9}$$

In (4.8) the upper inequality holds when (μ_0, l_0) are chosen to satisfy the constraints (3.18); the lower inequality corresponds to (3.22). In (4.9) μ_S is the shear component of the isotropic incompressible tensor $\hat{S}$. It is of the form

$$\mu_S \equiv \hat{S}_{1313} = \frac{1}{4\mu_0}\psi(l_0),$$

where the dimensionless function ψ is expressed in terms of the correlation coefficient h (see Appendix B):

$$\psi(l) = \frac{8}{5}\int_0^{+\infty} \mathrm{e}^{-2t} h(lt)\, t\, \mathrm{d}t. \tag{4.10}$$

Then, (4.8) gives the following bounds on μ_*:

$$\mu_* \underset{(\geqslant)}{\leqslant} \bar{\mu}(\mu_0, l_0), \tag{4.11}$$

where (μ_0, l_0) are chosen to satisfy the constraints (3.18) or (3.22), and the value for $\bar{\mu}$ is dependent upon the particular choice of values for (μ_0, l_0). The bounds (4.11) for the effective shear modulus μ_* are developed further in Section 4.1.

The relation (4.11) can also be regarded as the basis for a self-consistent estimation of μ_*, as suggested by Willis (1977) for conventional solids. In order to get a self-consistent estimate for μ_* bounds are also required for l_*. One can expect that (4.1) can also be used for the derivation of bounds for the effective length scale l_*. To estimate l_* we have to apply uniform curvature boundary conditions (and linear strain boundary conditions) to the macroelement V in (4.1). However, for volumes V large compared to the length scale, the linear strain terms will dominate the curvature terms in (4.1). We derive bounds for $\mu_* l_*^2$ in a speculative manner as follows. Choose

$\boldsymbol{\chi}_0$ to be constant and allow $\boldsymbol{\varepsilon}_0$ to vary linearly in $\mathbf{x}$, in a consistent manner such that $\bar{\boldsymbol{\varepsilon}}_0 = \mathbf{0}$. The bounds (4.5) apply with $\bar{\boldsymbol{\chi}}_0 = \boldsymbol{\chi}_0 = \text{constant}$ and $\bar{\boldsymbol{\varepsilon}}_0 = \mathbf{0}$ in V. Assume that the volume V is much less than l_0^3, such that we can write the strain energy of the comparison medium approximately as

$$\tilde{W}_0 = |V|\,[\mu_0\,\bar{\boldsymbol{\varepsilon}}_0 \cdot \bar{\boldsymbol{\varepsilon}}_0 + \mu_0 l_0^2\,\bar{\boldsymbol{\chi}}_0 \cdot \bar{\boldsymbol{\chi}}_0] = |V|\mu_0 l_0^2\,\boldsymbol{\chi}_0 \cdot \boldsymbol{\chi}_0. \tag{4.12}$$

Similarly, the strain energy of the composite is written approximately as

$$\tilde{W} = |V|\,[\mu_*\,\bar{\boldsymbol{\varepsilon}}_0 \cdot \bar{\boldsymbol{\varepsilon}}_0 + \mu_* l_*^2\,\bar{\boldsymbol{\chi}}_0 \cdot \bar{\boldsymbol{\chi}}_0] = |V|\mu_* l_*^2\,\boldsymbol{\chi}_0 \cdot \boldsymbol{\chi}_0.$$

Then, the Hashin–Shtrikman relations (4.5) reduce to

$$\mu_* l_*^2 \underset{(\geqslant)}{\leqslant} \bar{\mu}\bar{l}^2\,(\mu_0, l_0), \tag{4.13}$$

where

$$\bar{\mu}\bar{l}^2 = \left[\sum_{r=1}^{2} c_r\,\{1+4(\mu_r l_r^2 - \mu_0 l_0^2)\mu_Q\}^{-1}\right]^{-1} \sum_{j=1}^{2} c_j \mu_j l_j^2\,[1+4(\mu_j l_j^2 - \mu_0 l_0^2)\mu_Q]^{-1}. \tag{4.14}$$

Here μ_Q is the shear component of the tensor $\hat{Q}$:

$$\mu_Q \equiv \hat{Q}_{1313} = \frac{1}{4\mu_0 l_0^2}\eta\,(l_0),$$

and

$$\eta(l) = \frac{2}{5}\int_0^{+\infty} \mathrm{e}^{-2t}\,[1-h\,(lt)]\,t\,\mathrm{d}t$$

(see Appendix B).

We emphasize that the bounds (4.13) are speculative as our assumption that the representative volume $|V| \ll l_0^3$ is somewhat inconsistent with the notion that the volume V is representative of the composite.

The bounds for $\mu_* l_*^2$ are not developed further but will be used in a self-consistent scheme (Section 5.4). The bounds for μ_* are, however, reliable and we now propose a scheme for their optimization.

4.1. *Bounds for the effective shear modulus*

We seek explicit expressions for upper and lower bounds on the composite effective shear modulus μ_*. Combining (4.9) and (4.11) gives

$$\mu_* \underset{(\leqslant)}{\geqslant} \mu_2 + \frac{c_1\,(\mu_1 - \mu_2)}{1 + \dfrac{\mu_1 - \mu_2}{\mu_2 + \mu_0\left(\dfrac{1}{\psi(l_0)} - 1\right)}}. \tag{4.15}$$

The $(\geqslant)$ inequality in (4.15) holds provided

$$\mu_0 < \min\{\mu_1, \mu_2\} \quad \text{and} \quad \mu_0 l_0^2 < \min\{\mu_1 l_1^2, \mu_2 l_2^2\} \tag{4.16}$$

and, for an optimal choice of (μ_0, l_0) gives a lower bound on μ_*. Similarly, the $(\leqslant)$ inequality in (4.15) holds provided

$$\mu_0 > \max\{\mu_1, \mu_2\} \quad \text{and} \quad \mu_0 l_0^2 > \max\{\mu_1 l_1^2, \mu_2 l_2^2\} \tag{4.17}$$

and, for an optimal choice of (μ_0, l_0) gives an upper bound on μ_*.

On physical grounds the upper bound on μ_* cannot be less than the lower bound. This places the following restrictions on the admissible form of ψ:

(i) $\psi(l)$ must decrease monotonically with increasing l;

(ii) $\dfrac{1}{l^2}\left\{\dfrac{1}{\psi(l)} - 1\right\}$ must decrease monotonically with increasing l.

Note that these restrictions on ψ place restrictions on the correlation function h via (4.10).

Assume without loss of generality that $\mu_1 \geqslant \mu_2$. On making use of restrictions (i) and (ii) the optimal choice of (μ_0, l_0) to achieve a lower bound from (4.15) is $\mu_0 \to \mu_2$ and $\mu_0 l_0^2 \to \min\{\mu_1 l_1^2, \mu_2 l_2^2\}$. This gives

$$\mu_* \geqslant \mu_2 \left\{ 1 + \frac{c_1}{\dfrac{\mu_2}{\mu_1 - \mu_2} + \psi^- c_2} \right\}, \tag{4.18}$$

where

$$\psi^- = \psi(l^-), l^- = \min\left\{ l_2, \sqrt{\frac{\mu_1}{\mu_2}}\, l_1 \right\},$$

and $\psi(l)$ is related to the correlation coefficient $h(|\mathbf{z}|)$ via (4.10).

In similar fashion the optimal choice of (μ_0, l_0) to achieve an upper bound from (4.15) is $\mu_0 \to \mu_1$ and $\mu_0 l_0^2 \to \max\{\mu_1 l_1^2, \mu_2 l_2^2\}$. This gives

$$\mu_* \leqslant \mu_1 \left\{ 1 + \frac{c_2}{\dfrac{\mu_1}{\mu_2 - \mu_1} + \psi^+ c_1} \right\}, \tag{4.19}$$

with

$$\psi^+ = \psi(l^+), l^+ = \max\left\{ l_1, \sqrt{\frac{\mu_2}{\mu_1}}\, l_2 \right\}.$$

The formulae (4.18) and (4.19) demonstrate simple bounds of Hashin–Shtrikman type for a statistically isotropic two-phase composite in terms of volume fractions *and* an isotropic correlation function $h(|\mathbf{z}|)$ which accounts for the size effect. As a simple example we take

$$h(\mathbf{z}) = \mathrm{e}^{-|\mathbf{z}|/a}, \tag{4.20}$$

where a is a correlation length related, for example, to the typical grain size of each phase. The associated expression for $\psi(l) = \tilde{\psi}(l/a)$ is (see Appendix B)

$$\tilde{\psi}(z) = \frac{8}{5(z+2)^2}.$$

Related numerical results are presented in Section 6. In the conventional limit $l/a \to 0$, $\psi^+ = \psi^- = 2/5$ and the inequalities (4.18), (4.19) transform into the "classical" Hashin–Shtrikman (1963) bounds for incompressible solids.

Finally, if one of the phases is rigid ($\mu_1 \to \infty$), the upper bound becomes infinitely large but the lower bound degenerates to

$$\mu_- = \mu_2 \left\{ 1 + \frac{c_1}{c_2 \tilde{\psi}(l_2/a)} \right\}.$$

In the dilute limit ($c_1 \to 0$) for rigid reinforcement

$$\mu_- \sim \mu_2(1 + c_1 f_p^-),$$

where the lower bound for the strengthening parameter f_p

$$f_p^- \left(\frac{l}{a}\right) = \frac{1}{\tilde{\psi}(l/a)} \tag{4.21}$$

can be compared with the predictions of Fleck and Hutchinson (1993) (see Section 6).

Alternatively, the limit $\mu_2 \to 0$ corresponds to the case of voids filled with an incompressible liquid; then, the lower bound vanishes and the upper bound simplifies to

$$\mu_+ = \mu_1 \left\{ 1 - \frac{c_2}{1 - c_1 \tilde{\psi}(l_1/a)} \right\}.$$

In the limit of a dilute concentration of voids ($c_2 \to 0$), we have

$$\mu_+ \sim \mu_1(1 - c_2 f_v^+),$$

where

$$f_v^+ \left(\frac{l}{a}\right) = \frac{1}{1 - \tilde{\psi}(l/a)}. \tag{4.22}$$

This gives an upper bound for the softening parameter f_v.

5. SELF-CONSISTENT ESTIMATES

Two different self-consistent schemes are now developed. One of them follows the approach of Budiansky (1965) and Hill (1965), and assumes that a composite consists of a matrix containing an infinite number of randomly distributed, equi-sized spherical inclusions. The other is based on the above Hashin–Shtrikman procedure and follows Willis (1977) in finding an "optimal" comparison medium.

We develop first the approach for spherical inclusions by noting some formal similarities of the strain-gradient formulation with that of the wave propagation problem studied by Sabina and Willis (1988).

5.1. *Composite with spherical inclusions*

Let the spherical inclusions of radius a comprise phase number one of the composite and have volume concentration $c = c_1$, and the matrix comprise phase number two. Define $\mathbf{u}_*(\mathbf{x})$ as the average displacement field over an *ensemble* of realizations of inclusion distribution, at each point $\mathbf{x}$ of the composite. Then, $\mathbf{u}_*(\mathbf{x})$ can be represented by a combination of the averages $\mathbf{u}_1(\mathbf{x})$ and $\mathbf{u}_2(\mathbf{x})$, conditional on $\mathbf{x}$ belonging to an inclusion or to the matrix:

$$\mathbf{u}_* = c\mathbf{u}_1 + (1-c)\mathbf{u}_2.$$

Similar relations hold for the averaged fields $\boldsymbol{\varepsilon}_*$, $\boldsymbol{\chi}_*$, $\mathbf{s}_*$ and $\mathbf{m}_*$.

Following Hill (1965), the constitutive relations (2.20) are averaged via simple algebraic manipulations as follows:

$$\begin{aligned}\mathbf{s}_* &= 2\mu_2\boldsymbol{\varepsilon}_* + 2c(\mu_1-\mu_2)\boldsymbol{\varepsilon}_1,\\ \mathbf{m}_* &= 2\mu_2 l_2^2\boldsymbol{\chi}_* + 2c(\mu_1 l_1^2 - \mu_2 l_2^2)\boldsymbol{\chi}_1.\end{aligned} \tag{5.1}$$

Now define the effective composite parameters μ_* and l_* by

$$\mathbf{s}_* = 2\mu_*\boldsymbol{\varepsilon}_*, \quad \mathbf{m}_* = 2\mu_* l_*^2\boldsymbol{\chi}_*. \tag{5.2}$$

On combining (5.1) and (5.2) we get

$$\begin{aligned}\mu_*\boldsymbol{\varepsilon}_* &= \mu_2\boldsymbol{\varepsilon}_* + c\,(\mu_1-\mu_2)\boldsymbol{\varepsilon}_1,\\ \mu_* l_*^2\boldsymbol{\chi}_* &= \mu_2 l_2^2\boldsymbol{\chi}_* + c\,(\mu_1 l_1^2 - \mu_2 l_2^2)\boldsymbol{\chi}_1.\end{aligned} \tag{5.3}$$

These formulae provide equations for μ_* and l_* once the conditional averages $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\chi}_1$ have been obtained in terms of $\boldsymbol{\varepsilon}_*$ and $\boldsymbol{\chi}_*$. To obtain expressions for (μ_*, l_*) we take two trial fields in turn: (i) $\boldsymbol{\varepsilon}_* =$ constant and $\boldsymbol{\chi}_* = \mathbf{0}$, and (ii) $\boldsymbol{\chi}_* =$ constant and therefore $\boldsymbol{\varepsilon}_*$ is linear in $\mathbf{x}$.

The conditional averages $(\boldsymbol{\varepsilon}_1, \boldsymbol{\chi}_1)$ are the average values of strain and curvature at a point $\mathbf{x}$, assuming that $\mathbf{x}$ belongs to an inclusion. The basic prescription of the self-consistent method of Budiansky and Hill requires us to estimate $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\chi}_1$ in terms of the total effective field $(\boldsymbol{\varepsilon}_*, \boldsymbol{\chi}_*)$ as follows.

Consider an isolated representative inclusion which is centred at $\mathbf{x}'$ and contains our point of interest $\mathbf{x}$. Then $\mathbf{x}'$ is such that $|\mathbf{x}'-\mathbf{x}| < a$. The inclusion is embedded into the "effective medium" (with unknown effective parameters μ_* and l_*), and the field remote from the inclusion is specified as $(\boldsymbol{\varepsilon}_*(\mathbf{x}), \boldsymbol{\chi}_*(\mathbf{x}))$ (see Fig. 1).

To proceed, we estimate the average strain and curvature in the inclusion $(\bar{\boldsymbol{\varepsilon}}(\mathbf{x}'), \bar{\boldsymbol{\chi}}(\mathbf{x}'))$ for an assumed remote field $(\boldsymbol{\varepsilon}_*, \boldsymbol{\chi}_*)$. We adopt the strategy of calculating approximately the average strain $\bar{\boldsymbol{\varepsilon}}(\mathbf{x}')$ and average curvature $\bar{\boldsymbol{\chi}}(\mathbf{x}')$ in the inclusion. The conditional average $\boldsymbol{\varepsilon}_1$ is related to $\bar{\boldsymbol{\varepsilon}}(\mathbf{x}')$ by

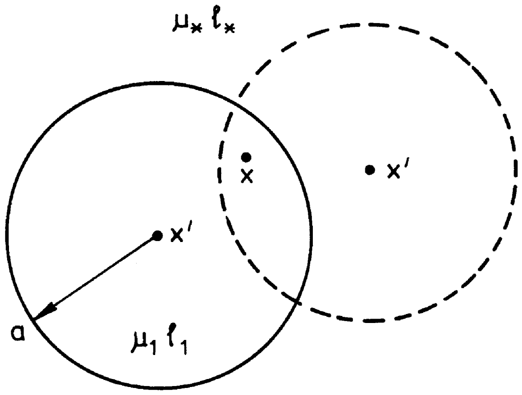

Fig. 1. An inclusion of radius a is embedded in the effective medium. A self-consistent estimation of $\mathbf{u}_1(\mathbf{x})$ averages the solution $\mathbf{u}(\mathbf{x},\mathbf{x}')$ over all possible inclusion centres such that the point $\mathbf{x}$ still belongs to the inclusion.

$$\varepsilon_1(\mathbf{x}) \approx \frac{1}{|\Omega|}\int_{\mathbf{x}'\in\Omega} \bar{\varepsilon}(\mathbf{x}')\,\mathrm{d}\mathbf{x}', \tag{5.4}$$

where $\Omega \equiv |\mathbf{x}'-\mathbf{x}| < a$. In similar manner χ_1 is given by

$$\chi_1(\mathbf{x}) \approx \frac{1}{|\Omega|}\int_{\mathbf{x}'\in\Omega} \bar{\chi}(\mathbf{x}')\,\mathrm{d}\mathbf{x}'. \tag{5.5}$$

We now consider the isolated inclusion problem in order to determine $\bar{\varepsilon}(\mathbf{x}')$ and $\bar{\chi}(\mathbf{x}')$.

5.2. *The isolated inclusion problem*

Let an infinite "effective" matrix of properties μ_*, l_* support a displacement field $\mathbf{u}_*(\mathbf{x})$ in the absence of an inclusion. Consider now a single spherical inclusion of radius a embedded into the matrix and centred at the point $\mathbf{x}'$ as shown in Fig. 1.

The total field both inside and outside the inclusion can be represented as

$$\mathbf{u} = \mathbf{u}_* + \tilde{\mathbf{u}},$$

where $\tilde{\mathbf{u}}$ is the additional displacement due to introduction of the inclusion. The system of integral equations for the solution of $\mathbf{u}$ is derived in the strain-gradient context in a standard way. For this purpose the matrix material can be treated as the comparison material, i.e. we take $\mu_0 = \mu_*$, $l_0 = l_*$, $\mathbf{u}_0 = \mathbf{u}_*$, $\boldsymbol{\varepsilon}_0 = \boldsymbol{\varepsilon}_*$ and $\boldsymbol{\chi}_0 = \boldsymbol{\chi}_*$. We introduce the polarizations $\boldsymbol{\pi}$ and $\boldsymbol{\beta}$ via

$$s_{ij} \equiv 2\mu_0\varepsilon_{ij} + \pi_{ij},$$
$$m_{ij} \equiv 2\mu_0 l_0^2 \chi_{ij} + \beta_{ij} \tag{5.6}$$

[cf. (3.4)]. Next, substitute (5.6) into the equilibrium statements (2.14) and (2.13). The resulting differential equation coincides with (3.8) and (3.9). The solution to our single problem follows from (3.13) as

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_0 - S^{\infty}\boldsymbol{\pi} - M^{\infty}\boldsymbol{\beta}, \tag{5.7}$$

$$\boldsymbol{\chi} = \boldsymbol{\chi}_0 - P^{\infty}\boldsymbol{\pi} - Q^{\infty}\boldsymbol{\beta}, \tag{5.8}$$

with kernels (4.3). For example, the operator $S^{\infty}\boldsymbol{\pi}$ reads

$$S^{\infty}\boldsymbol{\pi}(\mathbf{x}) = \int_{\Omega} S^{\infty}(\mathbf{x}-\mathbf{x}')\boldsymbol{\pi}(\mathbf{x}')\,\mathrm{d}\mathbf{x}'.$$

We now formulate a system of integral equations for the determination of $\boldsymbol{\pi}$ and $\boldsymbol{\beta}$ within the inclusion volume Ω.

By writing the constitutive law for the inclusion as $s_{ij} = 2\mu_1\varepsilon_{ij}$ and $m_{ij} = 2\mu_1 l_1^2\chi_{ij}$, we eliminate $\mathbf{s}$ and $\mathbf{m}$ from (5.6) to get

$$\boldsymbol{\varepsilon} = \tfrac{1}{2}(\mu_1-\mu_0)^{-1}\boldsymbol{\pi}, \quad \boldsymbol{\chi} = \tfrac{1}{2}(\mu_1 l_1^2 - \mu_0 l_0^2)^{-1}\boldsymbol{\beta}. \tag{5.9}$$

In combination with (5.8) this gives

$$\begin{aligned} &\tfrac{1}{2}(\mu_1-\mu_0)^{-1}\boldsymbol{\pi} + S^{\infty}\boldsymbol{\pi} + M^{\infty}\boldsymbol{\beta} = \boldsymbol{\varepsilon}_0, \\ &\tfrac{1}{2}(\mu_1 l_1^2 - \mu_0 l_0^2)^{-1}\boldsymbol{\beta} + P^{\infty}\boldsymbol{\pi} + Q^{\infty}\boldsymbol{\beta} = \boldsymbol{\chi}_0. \end{aligned} \tag{5.10}$$

Equivalently, the above integral equations can be stated in the form of a stationary principle:

$$\delta\Phi^{\infty}(\boldsymbol{\pi}, \boldsymbol{\beta}) = 0, \tag{5.11}$$

where

$$\Phi^{\infty}(\boldsymbol{\pi}, \boldsymbol{\beta}) \equiv \tfrac{1}{2}(\mu_1-\mu_0)^{-1}\langle\boldsymbol{\pi},\boldsymbol{\pi}\rangle + \tfrac{1}{2}(\mu_1 l_1^2 - \mu_0 l_0^2)^{-1}\langle\boldsymbol{\beta},\boldsymbol{\beta}\rangle + \Gamma^{\infty}(\boldsymbol{\pi},\boldsymbol{\beta}) - 2\langle\boldsymbol{\varepsilon}_0,\boldsymbol{\pi}\rangle - 2\langle\boldsymbol{\chi}_0,\boldsymbol{\beta}\rangle. \tag{5.12}$$

The combination of inner products Γ^{∞} is defined by (3.15) with S replaced by S^{∞} etc. The domain of integration for the inner product in (5.12) is the inclusion volume Ω. The equivalence of (5.10) and (5.11) is secured by the symmetry property

$$\langle\boldsymbol{\pi}, M^{\infty}\boldsymbol{\beta}\rangle = \langle\boldsymbol{\beta}, P^{\infty}\boldsymbol{\pi}\rangle$$

for arbitrary $\boldsymbol{\pi}$, $\boldsymbol{\beta}$, which can be derived directly from (4.3).

In the limit $l/a \to 0$, M^{∞}, P^{∞} and Q^{∞} vanish and (5.10) reduces to standard Eshelby (1957) theory. For the general case ($l/a \neq 0$), it is difficult to solve (5.10) exactly and a simple approximation of Galerkin type is used to solve (5.11) by *taking* $\boldsymbol{\pi}$ *and* $\boldsymbol{\beta}$ *to be constant over the inclusion.* Equations (5.10) are then solved "on the average" over the inclusion volume by equating the mean values of the right and left hand sides (cf. Sabina and Willis, 1988), giving

$$\begin{aligned} &\tfrac{1}{2}(\mu_1-\mu_0)^{-1}\boldsymbol{\pi} + \bar{S}\boldsymbol{\pi} + \bar{M}\boldsymbol{\beta} = \bar{\boldsymbol{\varepsilon}}_0(\mathbf{x}'), \\ &\tfrac{1}{2}(\mu_1 l_1^2 - \mu_0 l_0^2)^{-1}\boldsymbol{\beta} + \bar{P}\boldsymbol{\pi} + \bar{Q}\boldsymbol{\beta} = \bar{\boldsymbol{\chi}}_0(\mathbf{x}'). \end{aligned} \tag{5.13}$$

Here

$$\bar{S} \equiv \frac{1}{|\Omega|}\int_{\Omega}\int_{\Omega} S^{\infty}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}', \quad \bar{Q} \equiv \frac{1}{|\Omega|}\int_{\Omega}\int_{\Omega} Q^{\infty}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}' \tag{5.14}$$

with similar relations for $\bar{M}$ and $\bar{P}$; and

$$\bar{\boldsymbol{\varepsilon}}_0 \equiv \frac{1}{|\Omega|}\int_{\Omega} \boldsymbol{\varepsilon}_0(\mathbf{x})\,\mathrm{d}\mathbf{x}, \quad \bar{\boldsymbol{\chi}}_0 \equiv \frac{1}{|\Omega|}\int_{\Omega} \boldsymbol{\chi}_0(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{5.15}$$

Note that we have converted the system of integral equations (5.10) into a much simpler set of algebraic equations (5.13).

Simple analysis shows that $\bar{M} = \bar{P} = 0$ because $M^{\infty}(\mathbf{x})$ and $P^{\infty}(\mathbf{x})$ are odd functions [see (4.3)]. Therefore, the system (5.13) uncouples to the form

$$\boldsymbol{\pi} = [\tfrac{1}{2}(\mu_1-\mu_0)^{-1}I+\bar{S}]^{-1}\bar{\boldsymbol{\varepsilon}}_0,$$

$$\boldsymbol{\beta} = [\tfrac{1}{2}(\mu_1 l_1^2-\mu_0 l_0^2)^{-1}I+\bar{Q}]^{-1}\bar{\boldsymbol{\chi}}_0.$$

Isotropy and incompressibility dictate that only shear components are involved, and (5.9) gives

$$\bar{\varepsilon}_{ij}(\mathbf{x}') = [1+4(\mu_1-\mu_0)\tilde{\mu}_S]^{-1}\bar{\varepsilon}_{0ij}(\mathbf{x}'),$$

$$\bar{\chi}_{ij}(\mathbf{x}') = [1+4(\mu_1 l_1^2-\mu_0 l_0^2)\tilde{\mu}_Q]^{-1}\bar{\chi}_{0ij}(\mathbf{x}') \tag{5.16}$$

within the inclusion $\Omega(\mathbf{x}')$. Here $\tilde{\mu}_S = \bar{S}_{1313}$ and $\tilde{\mu}_Q = \bar{Q}_{1313}$ are the shear component of the tensors $\bar{S}$ and $\bar{Q}$ (Appendix B). The expressions (5.16) for $(\bar{\boldsymbol{\varepsilon}}, \bar{\boldsymbol{\chi}})$ are substituted into relations (5.4) and (5.5) in order to obtain the conditional averages $(\boldsymbol{\varepsilon}_1, \boldsymbol{\chi}_1)$.

5.3. *Self-consistent equations* (*spherical inclusions*)

The above prescription permits us to derive self-consistent expressions for the composite modulus μ_0 $(=\mu_*)$ and the composite length scale l_0 $(=l_*)$. In order to determine μ_0 and l_0 we must select appropriate trial fields $(\boldsymbol{\varepsilon}_0, \boldsymbol{\chi}_0)$.

The simplest option is to choose $\boldsymbol{\varepsilon}_0 = \text{const.}$ (and therefore $\boldsymbol{\chi}_0 \equiv 0$). Then, (5.3) and (5.16) give

$$\mu_0 = \mu_2 + c\,(\mu_1-\mu_2)[1+4(\mu_1-\mu_0)\tilde{\mu}_S(\mu_0, l_0)]^{-1}. \tag{5.17}$$

To derive a second equation we choose the curvature $\boldsymbol{\chi}_0(\mathbf{x})$ to be constant and the strain $\boldsymbol{\varepsilon}_0(\mathbf{x})$ to be linear in $\mathbf{x}$. The displacement field is written as

$$u_{0i}(\mathbf{x}) = \tfrac{1}{2}\gamma_{ijk}x_k x_j,$$

such that γ_{ijk} generates a constant curvature field.†

The relations (5.1), (5.2) and (5.16) then give

$$\mu_0 l_0^2 = \mu_2 l_2^2 + c\,(\mu_1 l_1^2-\mu_2 l_2^2)[1+4(\mu_1 l_1^2-\mu_0 l_0^2)\tilde{\mu}_Q(\mu_0, l_0)]^{-1}. \tag{5.18}$$

Making use of the explicit form for $\tilde{\mu}_S$ and $\tilde{\mu}_Q$ given in (B9) the formulae (5.17) and (5.18) finally transform to

† Incompressibility dictates that $\gamma_{iki}+\gamma_{iik} = 0$.

$$\mu_0 = \mu_2 + c\frac{\mu_1-\mu_2}{1+\dfrac{2}{5}\dfrac{\mu_1-\mu_0}{\mu_0}(1-\beta(2\,a/l_0))},$$

$$\mu_0 l_0^2 = \mu_2 l_2^2 + c\frac{\mu_1 l_1^2-\mu_2 l_2^2}{1+\dfrac{\mu_1 l_1^2-\mu_0 l_0^2}{10\mu_0 l_0^2}\beta(2\,a/l_0)}, \tag{5.19}$$

with $\beta(z)$ defined by (B10). Numerical values for μ_0 and l_0 can be extracted from the implicit system of equations (5.19) by numerical iteration (see Section 6). For $l/a \to 0$ the system uncouples and the first equation reduces to that of Hill (1965) and Budiansky (1965).

In the *dilute* limit ($c \to 0$), the system (5.19) becomes an explicit set of formulae for (μ_0, l_0) since μ_0 and l_0 on the right sides are replaced by μ_2 and l_2:

$$\mu_0 \cong \mu_2 + c\frac{\mu_1-\mu_2}{1+\dfrac{2}{5}\dfrac{\mu_1-\mu_2}{\mu_2}(1-\beta(2a/l_2))},$$

$$\mu_0 l_0^2 \cong \mu_2 l_2^2 + c\frac{\mu_1 l_1^2-\mu_2 l_2^2}{1+\dfrac{\mu_1 l_1^2-\mu_2 l_2^2}{10\mu_2 l_2^2}\beta(2a/l_2)}.$$

In particular, for a dilute concentration of rigid inclusions ($\mu_1 \to \infty$) the first formula gives

$$\mu_0 \cong \mu_2 + \tfrac{5}{2}c\frac{\mu_2}{1-\beta(2a/l_2)}. \tag{5.20}$$

In the other limit of voids filled with an incompressible fluid ($\mu_1 \to 0$) we have:

$$\mu_0 \cong \mu_2 - c\frac{5\mu_2}{3+2\beta(2a/l_2)}. \tag{5.21}$$

5.4. *Self-consistent equations based on the Hashin–Shtrikman procedure*

An alternative self-consistent approach follows Willis (1977, 1981) by choosing an "optimal" comparison medium. In the strain-gradient context this entails (i) replacing (μ_*, l_*) by (μ_0, l_0) in the left hand side of (4.11) and (4.13), and (ii) replacing the inequalities in (4.11), (4.13) by equalities, to give the following implicit equations for μ_0 and l_0:

$$\mu_0 = \bar{\mu}(\mu_0, l_0), \quad \mu_0 l_0^2 = \overline{\mu l^2}\,(\mu_0, l_0).$$

Then (4.9) and (4.14) become

$$\mu_0 = \left[\sum_{r=1}^{2} c_r \{1+4(\mu_r-\mu_0)\mu_S\}^{-1}\right]^{-1} \cdot \sum_{j=1}^{2} c_j\mu_j\,[1+4(\mu_j-\mu_0)\mu_S]^{-1},$$

$$\mu_0 l_0^2 = \left[\sum_{r=1}^{2} c_r \{1+4(\mu_r l_r^2-\mu_0 l_0^2)\mu_Q\}^{-1}\right]^{-1} \cdot \sum_{j=1}^{2} c_j\mu_j l_j^2\,[1+4(\mu_j l_j^2-\mu_0 l_0^2)\mu_Q]^{-1}. \quad (5.22)$$

We note that (5.22) can be manipulated into the same algebraic form as the relations (5.17) and (5.18) derived by the Budiansky–Hill method. The only difference is that μ_S and μ_Q in (5.22), and $\tilde{\mu}_Q$ and $\tilde{\mu}_S$ in (5.17) and (5.18) are given by different formulae. Explicit expressions are given in Appendix B for μ_S and μ_Q in the Hashin–Shtrikman procedure [for the correlation (4.20)] and for $\tilde{\mu}_Q$ and $\tilde{\mu}_S$ in the alternative self-consistent scheme. In the conventional limit $l/a \to 0$ both self-consistent approaches become equivalent, as has been discussed by Willis (1977).

6. NUMERICAL RESULTS

The results obtained in the preceding sections permit straightforward numerical implementation. The *bounds* were calculated explicitly from analytic formulae (4.18), (4.19) and (B8). The relevant *self-consistent* equations are (5.17) and (5.18) for both the Budiansky–Hill scheme and (with suitable modification) for the scheme based on the Hashin–Shtrikman procedure. They were solved by iteration for given values of c_1 and l, starting from the dilute limits and then by parameter tracking. The iteration scheme demonstrated rapid convergence.

As a first example the Hashin–Shtrikman bounds and the related self-consistent estimates have been calculated for a two-phase composite with $\mu_1/\mu_2 = 2$ and 5 and $l_1 = l_2 = 1$. The self-consistent estimate is compared with the Hashin–Shtrikman bounds in Figs 2 and 3. The effect of concentration c_1 on composite modulus is shown explicitly in Fig. 2a for $\mu_1/\mu_2 = 2$ and in Fig. 2b for $\mu_1/\mu_2 = 5$. Similarly, the effect of length scale l is displayed in Fig. 3a for $\mu_1/\mu_2 = 2$ and in Fig. 3b for $\mu_1/\mu_2 = 5$. Note that the composite stiffness increases with increasing l/a (see Fig. 3). In all cases the self-consistent estimates lie between the upper and lower Hashin–Shtrikman bounds, and in turn the Hashin–Shtrikman bounds lie between the elementary (Voigt and Reuss) bounds. Clearly, the Hashin–Shtrikman bounds are much closer together than the elementary bounds. When $l/a = 0$ the results reproduce the conventional Hashin–Shtrikman bounds and the self-consistent estimates of Budiansky and Hill.

The Budiansky–Hill self-consistent estimate (based on the "spherical inclusions" assumption) and the self-consistent estimate based on the Hashin–Shtrikman procedure [for the "exponential" correlation (4.20)] are compared in Fig. 4. We show the effect of volume concentration c_1 upon the effective shear modulus in Fig. 4a for the case $\mu_1/\mu_2 = 5$ and $l_1 = l_2 = l = a$. The effect of the non-dimensional length scale l/a upon effective shear modulus is given explicitly in Fig. 4b (again for $\mu_1/\mu_2 = 5$). It is noted from both Fig. 4a and b that the two self-consistent schemes give similar

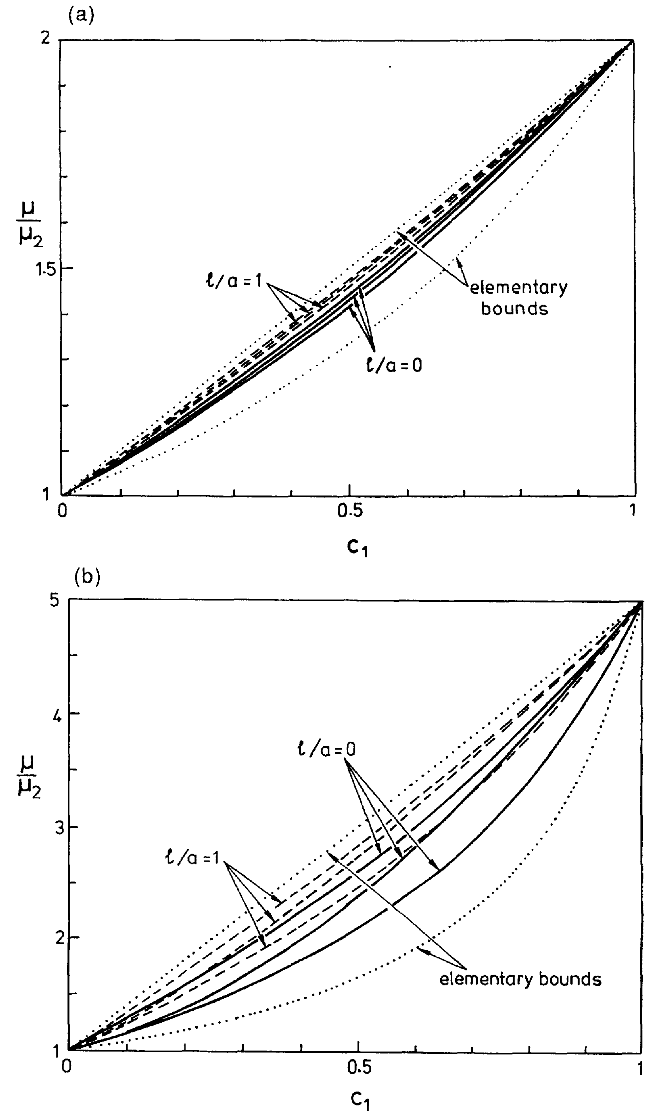

Fig. 2. Effect of inclusion concentration upon the Hashin–Shtrikman upper and lower bounds, and related self-consistent estimates. Results are shown for both $l/a = 0$ and $l/a = 1$, with $l_1 = l_2 = l$ and the correlation function (4.20). In (a) $\mu_1 = 2\mu_2$, and in (b) $\mu_1 = 5\mu_2$.

effective moduli, and that the effective moduli increase with increasing l/a. In fact, for the case $l/a = 0$, the two schemes give identical predictions [as observed by Willis (1977)]; the case $l/a = 0$ is included in Fig. 4a for comparison purposes. The effect of concentration c_1 upon the effective length scale l_0 is given in Fig. 4c, for $l_1 = 5l_2 = 5a$ and $\mu_1 = \mu_2$. Again, the two self-consistent approaches give almost identical predictions. As expected, l_0 converges to l_1 as $c_1 \to 1$ and l_0 converges to l_2 as $c_1 \to 0$. For intermediate values of c_1 the effective length scale l_0 is somewhat higher than the arithmetic mean value ("rule of mixtures").

Figure 5 shows the self-consistent estimate (5.20) for strengthening due to a *dilute* concentration of rigid spherical particles; for comparison the Hashin–Shtrikman lower bound (4.21) and the exact results of Fleck and Hutchinson (1993) are included. (The results of Fleck and Hutchinson apply directly to the constitutive law of the present paper upon appropriate re-scaling, as outlined in Appendix C.) The self-

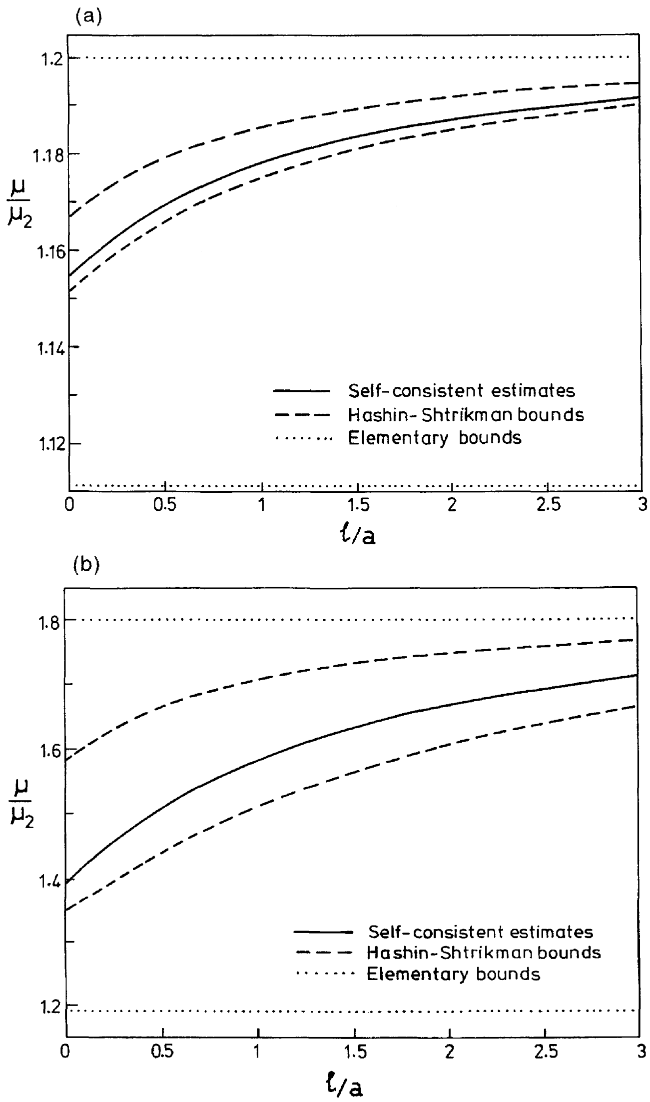

Fig. 3. Effect of ratio of length scales l/a upon the Hashin–Shtrikman upper and lower bounds, and related self-consistent estimates. Results are presented for $c_1 = 0.2$ and $l_1 = l_2 = l$. In (a) $\mu_1 = 2\mu_2$, and in (b) $\mu_1 = 5\mu_2$.

consistent scheme predicts slightly less strengthening than the exact result; this is due to the approximate nature of the solution to the isolated inclusion problem [see (5.16)].

7. CONCLUDING DISCUSSION

The constructions of the present paper demonstrate a simple generalization of the Hashin–Shtrikman and self-consistent techniques for two-phase linear composites, where the constitutive law of each phase involves couple stresses. The Hashin–Shtrikman bounds provide exact analytic formulae for bounding the effective shear modulus.

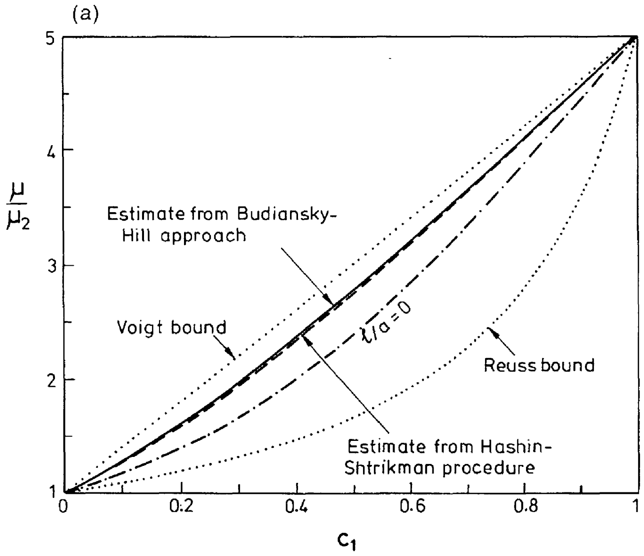

Fig. 4a. The self-consistent estimates as a function of concentration c_1 for $\mu_1 = 5\mu_2$, $l_1 = l_2 = l$ and $l/a = 1$. The self-consistent estimate based on the Hashin–Shtrikman procedure assumes the correlation function (4.20). For the case $l/a = 0$ the two self-consistent schemes give identical predictions.

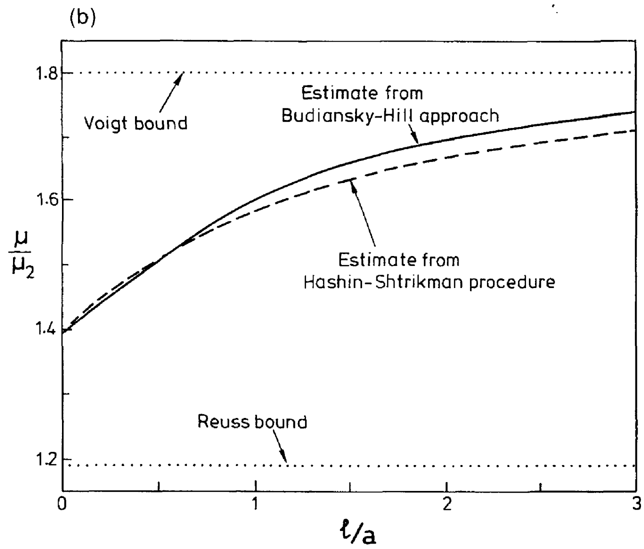

Fig. 4b. Comparison of Budiansky–Hill self-consistent estimates and the self-consistent estimates based on the Hashin–Shtrikman procedure, for $c_1 = 0.2$, $\mu_1 = 5\mu_2$ and $l_1 = l_2 = l$.

They are derived from piecewise constant polarizations. It is known that they are attainable bounds for the conventional solid where $l \to 0$ (see e.g. Milton, 1986). It is not clear to us whether the Hashin–Shtrikman bounds can be improved upon for a two-phase strain gradient composite by assuming a spatial variation in polarizations. The accuracy of the self-consistent scheme (based on the Budiansky–Hill approach) can be improved by a more detailed analysis of the isolated inclusion problem; this is not pursued here. It is likely that further technical development can also provide bounds for the effective length parameter l_* when the composite is subjected to macroscopic bending.

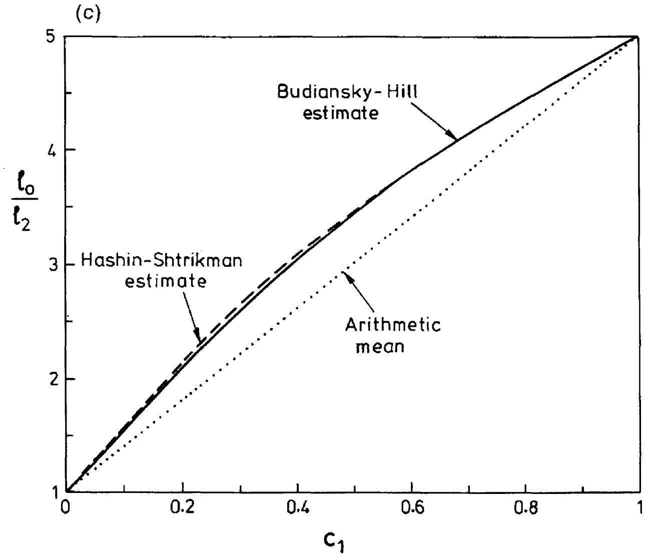

Fig. 4c. The self-consistent estimates of the effective length scale l_0 as a function of concentration c_1 for $\mu_1 = \mu_2$, $l_1 = 5l_2$ and $l_2/a = 1$.

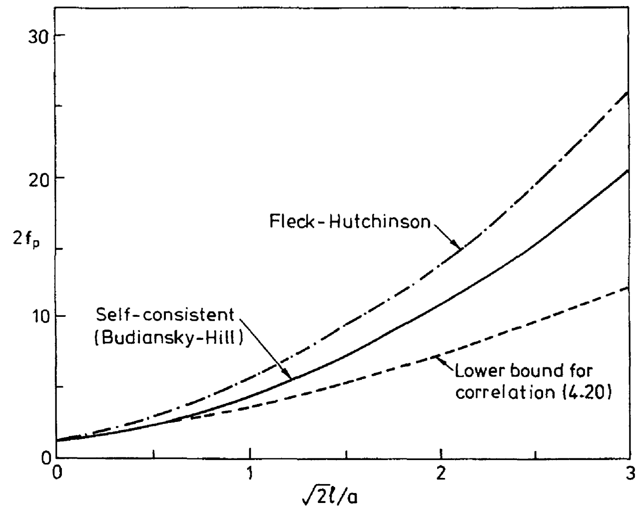

Fig. 5. Strengthening due to a dilute concentration of rigid spheres. The Hashin–Shtrikman lower bound (4.21) and the Budiansky–Hill self-consistent estimate are compared with the exact calculations of Fleck and Hutchinson (1993).

bounds for the effective length parameter l_* when the composite is subjected to macroscopic bending.

The schemes developed in the current paper can be generalized to other strain gradient media, such as those detailed by Mindlin (1965). Finally, the present work provides the *linear* background for estimating the response of *nonlinear* strain-gradient composites. This has been done recently for conventional nonlinear composites by Willis (1983, 1991), Talbot and Willis (1985) and Ponte-Castañeda (1991).

ACKNOWLEDGEMENTS

The authors are grateful for financial support from the U.S. Office of Naval Research, under contract number N00014-91-J-1916. The authors appreciate helpful discussions with Profs J. R. Willis and J. W. Hutchinson.

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APPENDIX A: OPERATORS RELATED TO THE GREEN'S FUNCTION

We start by deriving (3.14). We consider the right hand side of (3.14) and show that it can be rearranged into the left hand side. Recall from (3.13) that the linear operators S, M, P and Q are defined by

$$\tilde{\boldsymbol{\varepsilon}} = -S\boldsymbol{\pi} - M\boldsymbol{\beta}, \quad \tilde{\boldsymbol{\chi}} = -P\boldsymbol{\pi} - Q\boldsymbol{\beta}, \tag{A1}$$

where the displacement field $\tilde{\mathbf{u}}$ satisfies (3.9) and zero boundary conditions. Then $\Gamma(\boldsymbol{\pi}, \boldsymbol{\beta})$, as defined by (3.15), can be rearranged to the form

$$\Gamma(\boldsymbol{\pi},\boldsymbol{\beta}) = -\langle\boldsymbol{\pi},\tilde{\boldsymbol{\varepsilon}}\rangle - \langle\boldsymbol{\beta},\tilde{\boldsymbol{\chi}}\rangle,$$

where the expressions on the right hand side denote inner products as defined in (3.16). Now integrate by parts (twice) and use (3.7) and (3.9) to get

$$\Gamma(\boldsymbol{\pi}, \boldsymbol{\beta}) = \int_V (\pi_{ji,j}\tilde{u}_i + \beta_{pq,p}\tilde{\theta}_q)\,\mathrm{d}\mathbf{x}$$

$$= \int_V \{\pi_{ji,j} - \tfrac{1}{2}e_{pji}\beta_{pq,qj}\}\,\tilde{u}_i\,\mathrm{d}\mathbf{x} = \int_V \{-\mu_0\tilde{u}_{i,pp} + \tfrac{1}{4}\mu_0 l_0^2 e_{psi}e_{prj}\tilde{u}_{j,qqrs}\}\,\tilde{u}_i\,\mathrm{d}\mathbf{x}. \tag{A2}$$

Note that the equation inside the braces on the right hand side of (A2) equals $-L_0(\nabla)\tilde{\mathbf{u}}$ by the definition (3.7). Since

$$[L_0(\nabla)\tilde{\mathbf{u}} - L_0(\nabla)\mathbf{u}]_i = -[L_0(\nabla)\mathbf{u}_0]_i = \sigma^0_{\mathrm{h},i}$$

from (3.10), the relation (A2) can be rewritten as

$$\Gamma(\boldsymbol{\pi}, \boldsymbol{\beta}) = \int_V \{-\mu_0 u_{i,pp} + \tfrac{1}{4}\mu_0 l_0^2 e_{psi}e_{prj}u_{j,qqrs}\}\,\tilde{u}_i\,\mathrm{d}\mathbf{x}. \tag{A3}$$

Integration by parts of (A2) gives

$$\Gamma(\boldsymbol{\pi}, \boldsymbol{\beta}) = 2\mu_0\langle\tilde{\boldsymbol{\varepsilon}},\tilde{\boldsymbol{\varepsilon}}\rangle + 2\mu_0 l_0^2\langle\tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{\chi}}\rangle \tag{A4}$$

and the same manipulation on (A3) yields

$$\Gamma(\boldsymbol{\pi}, \boldsymbol{\beta}) = 2\mu_0\langle\boldsymbol{\varepsilon}, \tilde{\boldsymbol{\varepsilon}}\rangle + 2\mu_0 l_0^2\langle\boldsymbol{\chi}, \tilde{\boldsymbol{\chi}}\rangle. \tag{A5}$$

To complete the derivation of (3.14) we combine (A4) and (A5) with the definition of $\tilde{w}_0 = \mu_0\langle\boldsymbol{\varepsilon}_0, \boldsymbol{\varepsilon}_0\rangle + \mu_0 l_0^2\langle\boldsymbol{\chi}_0, \boldsymbol{\chi}_0\rangle$, and thereby obtain the left hand side of (3.14), since $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_0 + \tilde{\boldsymbol{\varepsilon}}$ and $\boldsymbol{\chi} = \boldsymbol{\chi}_0 + \tilde{\boldsymbol{\chi}}$.

To derive translation-invariant approximations for the operators S, M, P and Q we follow the strategy of Willis (1977, 1981). A sketch of the derivation follows. The infinite body Green's function $G_{ij}(\mathbf{x}-\mathbf{x}')$ is defined from (3.7) and (3.9) as

$$\mu_0 G_{ik,qq}(\mathbf{x}-\mathbf{x}') - \tfrac{1}{4}\mu_0 l_0^2 e_{psi}e_{prj}G_{jk,qqrs}(\mathbf{x}-\mathbf{x}') + \Lambda_{k,i} + \delta_{ik}\delta(\mathbf{x}-\mathbf{x}') = 0, \tag{A6}$$

where $\Lambda_k(\mathbf{x}-\mathbf{x}')$ is the equilibrium hydrostatic stress field due to a point load. Incompressibility implies that $G_{ij,i} \equiv 0$.

Consider again the displacement field $\tilde{\mathbf{u}}$ which satisfies (3.9) for some given $\boldsymbol{\pi}$ and $\boldsymbol{\beta}$ with $\tilde{\mathbf{u}} = \tilde{\boldsymbol{\theta}} = \mathbf{0}$ on the boundary. Multiplying (A6) by $\tilde{u}_i(\mathbf{x})$ and integrating over V with respect to $\mathbf{x}$, we obtain for arbitrary constants $\bar{\pi}_{ij}$ and $\bar{\beta}_{ij}$,

$$\tilde{u}_k(\mathbf{x}') = -\int_V \{G_{ik,j}(\mathbf{x}-\mathbf{x}')[\pi_{ji}(\mathbf{x})-\bar{\pi}_{ji}]\,\mathrm{d}\mathbf{x}+\tfrac{1}{4}e_{pji}G_{ik,qj}(\mathbf{x}-\mathbf{x}')[\beta_{pq}(\mathbf{x})-\bar{\beta}_{pq}]\}\,\mathrm{d}\mathbf{x}$$

$$-\int_S G_{ik}(\mathbf{x}-\mathbf{x}')\,T_i(\mathbf{x})\,\mathrm{d}s(\mathbf{x})+\int_S \tfrac{1}{4}e_{pji}G_{ik,j}(\mathbf{x}-\mathbf{x}')\,R_p(\mathbf{x})\,\mathrm{d}s(\mathbf{x}), \quad \text{(A7)}$$

where

$$T_i(\mathbf{x}) \equiv [\pi_{ji}(\mathbf{x})-\bar{\pi}_{ji}+2\mu_0\tilde{\varepsilon}_{ij}]\,n_j,$$

$$R_p(\mathbf{x}) \equiv [\beta_{pq}(\mathbf{x})-\bar{\beta}_{pq}+2\mu_0 l_0^2\tilde{\chi}_{pq}]\,n_q,$$

and $\mathbf{n}$ is the outward unit normal.

Next, let $\bar{\boldsymbol{\pi}}$ and $\bar{\boldsymbol{\beta}}$ be the mean values of polarizations $\boldsymbol{\pi}$ and $\boldsymbol{\beta}$ over V. Assume that the polarizations oscillate rapidly about these mean values. The reasonings of Willis (1977, 1981) can be adopted here as follows.

(i) Consider (A7) where $\mathbf{x}'$ is taken to lie on the boundary. Then the right hand side of (A7) vanishes due to zero displacement on the boundary.

(ii) Differentiate (A7) to transform $\tilde{u}_k$ into $\tilde{\theta}_r = \frac{1}{2}e_{rsk}\tilde{u}_{k,s}$ and take $\mathbf{x}'$ to lie on the boundary. Then the right hand side again vanishes due to zero rotation on the boundary.

(iii) Regard the resulting system of two equations as integral equations for $\mathbf{T}$ and $\mathbf{R}$. Since the volume integrals contain terms $(\boldsymbol{\pi}-\bar{\boldsymbol{\pi}})$ and $(\boldsymbol{\beta}-\bar{\boldsymbol{\beta}})$ which rapidly oscillate about zero, it is plausible to suppose that the solution $(\mathbf{T}, \mathbf{R})$ also oscillates about zero.

(iv) Granting this, the surface integral terms in (A7) are only significant when $\mathbf{x}'$ is in a "boundary layer" close to S. The approximation in (4.2) is to neglect the contribution from the surface integrals of (A7).

The operators S, M, P and Q in (A1) relate the polarizations to the strain field $\tilde{\boldsymbol{\varepsilon}}$ and to the curvature field $\tilde{\boldsymbol{\chi}}$. We obtain these operators by converting the expression (A7) for the displacement field $\tilde{\mathbf{u}}(\mathbf{x})$ into the corresponding expressions for the strain and curvature fields by differentiation.

APPENDIX B: EXPLICIT FORMULAE FOR THE GREEN'S FUNCTION AND FOR THE KERNEL FUNCTIONS S^∞ AND Q^∞

To derive an explicit formula for the infinite body Green's function defined by (A6) we use the method of Willis (1980) based on "plane wave" decomposition of the three-dimensional Dirac delta function $\delta(\mathbf{x})$. A useful property of $\delta(\mathbf{x})$ has been given by Gel'fand and Shilov (1964),

$$\delta(\mathbf{x}) = -\frac{1}{8\pi^2}\int_{|\xi|=1} \delta''(\boldsymbol{\xi}\cdot\mathbf{x})\,\mathrm{d}s(\boldsymbol{\xi}),$$

where δ'' is the second derivative of the one-dimensional delta function. This motivates an attempt to find $G_{ij}(\mathbf{x})$ in terms of the decomposition

$$G_{ij}(\mathbf{x}) = \frac{1}{8\pi^2\mu}\int_{|\xi|=1} F(\boldsymbol{\xi}\cdot\mathbf{x};\boldsymbol{\xi})\,\mathrm{d}s(\boldsymbol{\xi}),$$

where the function F_{ij} is to be found. Note that F_{ij} is taken to depend upon $\mathbf{x}$ only via the scalar quantity $(\boldsymbol{\xi}\cdot\mathrm{x})$, but is also assumed to be a function of $\boldsymbol{\xi}$.

Substitution of the above expressions for $\delta(\mathbf{x})$ and $G_{ij}(\mathbf{x})$ into (A6) gives

$$\int_{|\xi|=1} [F''_{ik} - \tfrac{1}{4}l^2 e_{psi}e_{prj}\xi_r\xi_s F^{(\mathrm{IV})}_{ij} + \xi_i\hat{\Lambda}_k - \xi_{ik}\delta''] \,\mathrm{d}s\,(\xi) = 0; \tag{B1}$$

Λ_k is written in terms of its transform $\hat{\Lambda}_k$, where

$$\Lambda_k(\mathbf{x}) = \frac{1}{8\pi^2\mu}\int_{|\xi|=1} \hat{\Lambda}_k(\boldsymbol{\xi}\cdot\mathbf{x};\xi)\,\mathrm{d}s(\xi).$$

The primes in (B1) denote differentiations with respect to the scalar argument $y \equiv \boldsymbol{\xi}\cdot\mathbf{x}$. The incompressibility condition implies that $\xi_j F_{jk} = 0$. We seek a solution F_{ij} such that the integrand of the above equation vanishes identically. Note also that (B1) is simplified using the identity

$$e_{psi}e_{prj}\xi_r\xi_s = \delta_{ij} - \xi_i\xi_j$$

for arbitrary $|\xi| = 1$.

Elementary analysis of the above relations coupled with the requirement for G (and therefore for F) to tend to zero at infinity gives

$$F(\boldsymbol{\xi}\cdot\mathbf{x}) = \frac{1}{l}(\delta_{ij} - \xi_i\xi_j)\exp\left\{-\frac{2}{l}|\boldsymbol{\xi}\cdot\mathbf{x}|\right\},$$

$$G_{ij}(\mathbf{x}) = \frac{1}{8\pi^2\mu l}\int_{|\xi|=1} (\delta_{ij} - \xi_i\xi_j)\exp\left\{-\frac{2}{l}|\boldsymbol{\xi}\cdot\mathbf{x}|\right\}\mathrm{d}s\,(\xi). \tag{B2}$$

We note that the representations (B2) for an incompressible *strain gradient* medium are formally similar to those of Willis [1980, formula (3.10)] for time-harmonic "conventional" *dynamics* (for the case of an incompressible isotropic medium). Specifically, the dynamic formula of Willis for the Green's function $G^{\mathrm{d}}\,(\mathbf{x},\lambda)$ depends on $\lambda = \omega^{-1}\sqrt{\mu/\rho}$, where ρ is density, ω is the angular frequency, and $2\pi\lambda$ is the wavelength of a shear wave. In the low-frequency limit $\lambda \to \infty$ the formula of Willis transforms to that for statics

$$G^{\mathrm{s}}_{ij}(\mathbf{x}) = \frac{1}{8\pi^2\mu l}\int_{|\xi|=1} (\delta_{ij} - \xi_i\xi_j)\delta(\boldsymbol{\xi}\cdot\mathbf{x})\,\mathrm{d}s\,(\xi).$$

In analogous fashion, the Green's function for the strain gradient solid transforms to the conventional static result when l tends to *zero*. The general *exact* correspondence between $G\,(\mathbf{x},l)$ and $G^{\mathrm{d}}\,(x;\lambda)$ for *finite* l and λ is

$$G_{ij}(\mathbf{x},l) = -G^{\mathrm{d}}_{ij}\left(\mathbf{x};-i\frac{l}{2}\right) + G^{\mathrm{s}}_{ij}(\mathbf{x}). \tag{B3}$$

Thus, we may interpret the strain gradient Green's function in terms of that for dynamics with an *imaginary* wavelength. The above simple connection permits us to use existing methods, developed for dynamics problems, for the strain gradient composites. See, for example, Sabina and Willis (1988), Sabina *et al.* (1993), and Smyshlyaev *et al.* (1993a,b).

The integral over the unit sphere in (B2) is easily calculated analytically leading to an explicit formula for $G_{ij}(\mathbf{x})$ similar to that derived by Mindlin (1965) for more general strain gradient media. This explicit formula is also derived from (B3) and an explicit representation for G^{d} [see e.g. Sabina and Willis (1988), formula (B.1)]:

$$G_{ij}(\mathbf{x},l) = \frac{1}{8\pi\mu}\left\{\delta_{ij}\frac{1-2\,\mathrm{e}^{-2|\mathbf{x}|/l}}{|\mathbf{x}|} + \frac{x_i x_j}{|\mathbf{x}|^3} - \tfrac{1}{2}l^2\frac{\partial^2}{\partial x_i\,\partial x_j}\left[\frac{1-\mathrm{e}^{-2|\mathbf{x}|/l}}{|\mathbf{x}|}\right]\right\}.$$

The formula (B2) provides the kernels S^∞, M^∞, P^∞ and Q^∞ defined in (4.3). For example,

$$S^{\infty}_{ijkl}(\mathbf{x}) = (2\pi^2\mu)^{-1}\int_{|\xi|=1}(-\delta_{ik}+\xi_i\xi_k)\xi_j\xi_{l|(ij)(kl)}\{-l^{-2}\delta(\xi\cdot\mathbf{x})+l^{-3}\,\mathrm{e}^{-2|\xi\cdot\mathbf{x}|/l}\}\,\mathrm{d}s(\xi) \quad \text{(B4)}$$

and

$$Q^{\infty}_{ijkl}(\mathbf{x}) = (8\pi^2\mu)^{-1}\int_{|\xi|=1}(-\delta_{ik}+\xi_i\xi_k)\xi_j\xi_{l|(ij)(kl)}$$
$$\times\{-l^{-4}\delta(\xi\cdot\mathbf{x})+l^{-5}\,\mathrm{e}^{-2|\xi\cdot\mathbf{x}|/l}+\tfrac{1}{4}l^{-2}\delta''(\xi\cdot\mathbf{x})\}\,\mathrm{d}s(\xi). \quad \text{(B5)}$$

In the case of elastodynamics, a kernel S^{d} $(\mathbf{x},\lambda)$ is related to G^{d} in the same way that S^{∞} $(\mathbf{x},l)$ is related to G_{ij} $(\mathbf{x},l)$. The transformation (B3) provides the connection between S^{∞} and S^{d}, and Q^{∞} and S^{d}:

$$S^{\infty}_{ijkl}(\mathbf{x},l) = -S^{\mathrm{d}}_{ijkl}\left(\mathbf{x};-i\frac{l}{2}\right)+S^{\mathrm{s}}_{ijkl}(\mathbf{x}),$$

$$Q^{\infty}_{ijkl}(\mathbf{x},l) = \tfrac{1}{4}l^{-2}S^{\mathrm{d}}_{ijkl}\left(\mathbf{x};-i\frac{l}{2}\right). \quad \text{(B6)}$$

The kernel functions S^{∞} and Q^{∞} appear in (4.6) in order to calculate the tensors $\hat{S}$ and $\hat{Q}$ in the Hashin–Shtrikman scheme. The fourth order tensors $\hat{S}$ and $\hat{Q}$ are isotropic in nature, and their shear components μ_S and μ_Q may be written as

$$\mu_S = \hat{S}_{1313}, \quad \mu_Q = \hat{Q}_{1313}. \quad \text{(B7)}$$

Expressions for μ_S and μ_Q follow from (B4), (B5) and (4.6):

$$\mu_S = \frac{1}{4\mu_0}\psi(l_0), \quad \mu_Q = \frac{1}{4\mu_0 l_0^2}\eta(l_0),$$

where

$$\psi(l) = \frac{8}{5}\int_0^{+\infty}\mathrm{e}^{-2t}h(lt)t\,\mathrm{d}t, \quad \eta(l) = \frac{2}{5}\int_0^{+\infty}\mathrm{e}^{-2t}[1-h(lt)]t\,\mathrm{d}t.$$

In particular, for the exponentially decaying correlation function (4.20), $\psi(l) = \tilde{\psi}(l/a)$, $\eta(l) = \tilde{\eta}(l/a)$, and

$$\tilde{\psi}(z) = \frac{8}{5(z+2)^2}, \quad \tilde{\eta}(z) = \frac{z(z+4)}{10(z+2)^2}. \quad \text{(B8)}$$

Observe that the explicit formula (B8) gives the limiting value $\psi = 2/5$ in the conventional limit $z = l_0/a \to 0$. Then, (4.18) and (4.19) reduce to "classical" Hashin–Shtrikman theory for the conventional solids.

In the Budiansky–Hill self-consistent scheme, the kernels $S^{\infty}(\mathbf{x},l)$ and $Q^{\infty}(\mathbf{x},l)$ are used to derive the tensors $\bar{S}$ and $\bar{Q}$ via (5.14). The moduli $\tilde{\mu}_S$ and $\tilde{\mu}_Q$, used in the self-consistent estimates (5.17) and (5.18), are the shear moduli of $\bar{S}$ and $\bar{Q}$, for example,

$$\tilde{\mu}_S = \bar{S}_{1313}, \quad \tilde{\mu}_Q = \bar{Q}_{1313}.$$

Explicit expressions may be obtained for $\tilde{\mu}_S$ and $\tilde{\mu}_Q$ directly from (5.14), (B4) and (B5), or from the connection (B6) and explicit formulae of Sabina and Willis (1988) as

$$\tilde{\mu}_S = \frac{1}{10\mu_0}\left\{1-\beta\left(2\frac{a}{l}\right)\right\},$$

$$\tilde{\mu}_Q = \frac{1}{40\,\mu_0 l_0^2}\beta\left(2\frac{a}{l}\right), \tag{B9}$$

where

$$\beta(z) = \frac{3(1+z)}{z^3}\,\mathrm{e}^{-z}\,(z\cosh z - \sinh z). \tag{B10}$$

APPENDIX C: UNSYMMETRIC CONSTITUTIVE RELATIONS

Consider a solid which supports an antisymmetric couple stress $\mathbf{m}^{\mathrm{a}}$ in addition to a symmetric couple stress, $\mathbf{m}^{\mathrm{s}}$. The constitutive law reads

$$m_{ij}^{\mathrm{s}} = 2\mu l_{\mathrm{s}}^2 \chi_{ij}^{\mathrm{s}}, \quad m_{ij}^{\mathrm{a}} = 2\mu l_{\mathrm{a}}^2 \chi_{ij}^{\mathrm{a}}.$$

For this solid, moment equilibrium implies

$$\tau_{ij} = -\mu(l_{\mathrm{s}}^2 + l_{\mathrm{a}}^2) e_{ijk} \chi_{pi,p}^{\mathrm{s}},$$

and the governing differential equation (3.10) in $\mathbf{u}$ becomes, via (3.7),

$$\mu_0 u_{i,pp} - \tfrac{1}{4}\mu_0 (l_{\mathrm{s}}^2 + l_{\mathrm{a}}^2)\, e_{psi} e_{prj} u_{j,qqrs} + \sigma_{\mathrm{h},i} = 0. \tag{C1}$$

Thus, the governing differential equation (3.10) for the symmetric solid is brought into agreement with (C1) for the more general solid by the simple transformation

$$l_{\mathrm{s}} \to \sqrt{l_{\mathrm{s}}^2 + l_{\mathrm{a}}^2}. \tag{C2}$$

Further, the boundary conditions for the case of *rigid* inclusions in the symmetric couple stress medium survive this transformation: $\mathbf{u} = \boldsymbol{\theta} = \mathbf{0}$ on the boundary of the inclusions. For non-rigid inclusions the connection is more complicated because the transformation (C1) fails to satisfy the interface conditions.

We conclude that the displacement field for the rigid particle problem in a symmetric couple stress solid (with length scale l_{s}) holds immediately for the more general solid (with length scales l_{s} and l_{a}) by making the transformation (C2). This argument enables us to make use of the existing solution of Fleck and Hutchinson (1993) with unsymmetric curvature ($l_{\mathrm{s}} = l_{\mathrm{a}}$) for a dilute concentration of rigid inclusions by appropriate re-scaling of l (see Fig. 5).