Steady-state constitutive relationship for idealised asphalt mixes

V.S. Deshpande, D. Cebon *

Department of Engineering, Cambridge University, Trumpington Street, Cambridge CB2 1PZ, UK

Received 28 October 1998

Abstract

A constitutive model for the steady-state deformation of idealised asphalt mixes is developed with the long term view to understanding the permanent deformation (rutting) of flexible pavements under vehicle loads. Triaxial compression tests are conducted on idealised asphalt mixes. Both volumetric and deviatoric strains are measured. The specimens are observed to dilate under compressive stresses and the deformation behaviour is seen to be dependent on the mean as well as the deviatoric stresses. A simple model for the overall steady-state behaviour of idealised mixes is developed using a “shear box” analogy from soil mechanics and theories for the deformation of nonlinear viscous composites. This constitutive relationship is dependent on the mean as well as deviatoric stresses and accounts for the dilation of the mixes under compressive loading. Predictions of the model are found to agree well with experimental measurements. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

The purpose of a pavement is to carry traffic safely, conveniently and economically throughout its design life. There are several reasons why a pavement may cease to fulfil its function. The most common forms of structural failure are permanent deformation (rutting) and fatigue (cracking).

The development of a model which explains the rutting of flexible pavements relies, at least in part, on understanding the constitutive behaviour of asphalt or bituminous mixes which are used in the upper layers of pavements. The majority of constitutive models for bituminous mixes are empirical and consider only uniaxial stress states (usually compressive stresses) (Deshpande and Cebon, 1998). However, in order to study the response of a pavement under a wheel load, an understanding of the overall constitutive behaviour of asphalt is needed.

Nijboer (1948) characterised bituminous mixes using soil mechanics theories. He distinguished between the behaviour of various mixes under triaxial loading using the angle of friction $\phi$ and the cohesion $c$. Huschek (1985) extended the work of Nijboer (1948) and proposed a model which separated the total deformation resistance of the mix into three components: initial resistance, internal friction due to the aggregate and the viscous resistance due to the bitumen. Each mechanism was modelled separately and then assembled to give an overall model describing the axial deformation behaviour of a mix under triaxial loading. However, his model needed to be calibrated by
performing complex experiments and only modelled the linear behaviour of asphalt.

Huang (1967) showed that there were certain important limitations in using linear viscoelastic models to describe the mix behaviour under triaxial loading. In particular, he observed that dilation occurred under applied axisymmetric compressive loading, and that the deformation behaviour depended on the confining pressure as well as the deviatoric stress. Since the discrepancies between linear viscoelastic models and experimental observations became more significant with increasing strains, he concluded that elastic and viscoelastic theories could only be used under transient loads of short durations and were not applicable for static wheel loads.

Brown and Cooper (1980, 1984) extensively studied the behaviour of bituminous mixes through triaxial tests and showed that the mix behaviour was a function of the hydrostatic as well as deviatoric stresses. Low et al. (1993) attempted to model the Marshall test using a plasticity model based on the Drucker–Prager yield criteria and the parameters $\phi$ and $c$. The rate dependence of the deformation was not considered, so all the results corresponded to a particular loading rate only.

The micro-structure of bituminous mixes can be used to understand their properties qualitatively as well as quantitatively. Notable such attempts include the model by van der Poel (1958) later named as the three phase model by Christensen and Lo (1979), the bitumen film creep model (Hills, 1973) and discrete element model (Rotthenburg et al., 1992).

The deformation behaviour of bituminous mixes is qualitatively well-understood through the various experimental studies performed over the last 50 years. A number of important conclusions regarding the deformation properties of bituminous mixes can be drawn from the available literature:

1. While bituminous mixes show linear viscoelastic behaviour at small strains, they are, in general, nonlinear materials. The nonlinear behaviour becomes more prominent at large strains (Monosmith and Secor, 1962; Pagen, 1965; Monosmith et al., 1966; Pagen, 1968).
2. The deformation of bituminous mixes is loading rate and temperature dependent. This dependence is the same as that of pure bitumen (van der Poel, 1955).
3. The deformation behaviour is a function of both the hydrostatic and deviatoric stress states (Huang, 1967).
4. The mixes dilate, even under purely deviatoric stresses (Huang, 1967).

There is no single model which is capable of explaining all the above observations in a consistent manner. A number of empirical models which approximate the behaviour of the mixes under specific loading conditions are available (e.g., Heukelom and Herrin, 1964; Brown and co-workers, 1992). These models have been reasonably successful in ranking mixes for paving purposes. However, their quantitative application to the analysis of pavements always requires large “calibration factors”.

To develop more durable asphalts suited to particular purposes, a better understanding of the fundamental mechanics of deformation and failure is needed. The various empirical and semi-empirical models developed to date do not provide an insight into these fundamentals. A micro-mechanical model for the overall constitutive behaviour of idealised bituminous mixes is described in this paper as a first step to understanding the behaviour of bituminous mixes used in pavements.

2. Triaxial experiments on idealised bituminous mixes

Creep tests on “idealised” bituminous mixes under various compressive triaxial stress states are described in this section. Preliminary conclusions of the possible mechanisms controlling the deformation of these mixes are drawn from the experimental observations.
2.1. Mix specification

2.1.1. Pure bitumen
Cheung and Cebon (1997) developed mathematical models for the deformation behaviour of a 50 pen \(^2\) bitumen over a wide range of temperatures and strain rates. For temperatures above approximately \(-10^\circ\text{C}\) they found that the steady-state deformation behaviour of bitumen could be described by the “Modified Cross Model” (Cheung and Cebon, 1997):

\[
\frac{\sigma}{\sigma_o} = \frac{\dot{\varepsilon}}{\dot{\varepsilon}_o} \left( \frac{1}{1 + (\dot{\varepsilon}/\dot{\varepsilon}_o)^n} \right),
\]

where \(n_c, \sigma_o, \dot{\varepsilon}_o\) are material constants for bitumen. They also found that the temperature dependence of bitumen was activation energy controlled (following an Arrhenius equation) at low temperatures \((T < 20\,^\circ\text{C}\) for their bitumen) and free volume controlled (WLF equation) at higher temperatures \((T > 20\,^\circ\text{C}\),

\[
\frac{\dot{\varepsilon}_o}{\dot{\varepsilon}_o} = \exp \left( \frac{Q_c}{RT} \right), \quad T \leq T_d,
\]

\[
\log \left( \frac{\dot{\varepsilon}_s}{\dot{\varepsilon}_o} \right) = -a_1(T - T_s) - a_2(T - T_d), \quad T \geq T_d.
\]

The values of the material constants \(\dot{\varepsilon}_o, \dot{\varepsilon}_s, Q_c, a_1, a_2, T_s, T_d, \sigma_o\) and \(n_c\) may be found in Cheung and Cebon (1997). The steady-state stress versus strain rate relationship (“deformation map”) for the bitumen tested by Cheung and Cebon (1997) is shown in Fig. 1. It shows that at \(20^\circ\text{C}\) the behaviour of bitumen can be approximated as linear viscous below \(\tau = 1 \times 10^5\) Pa, and power-law viscous with \(n = 2.3\) above \(\tau = 1 \times 10^5\) Pa. After consideration of the behaviour in shear, Cheung and Cebon (1997) concluded that in the power-law creep region, the constitutive behaviour of bitumen can be approximated as a von Mises material,

\[
\frac{\dot{\varepsilon}_{ij}}{\dot{\varepsilon}_o} = \frac{3}{2} \left( \frac{\Sigma_c}{\sigma_o} \right)^{n-1} \frac{\Sigma^\prime_{ij}}{\sigma_o},
\]

where \(\dot{\varepsilon}_o\) is the reference strain rate, \(\sigma_o\) the reference stress, \(\Sigma_c\) the Von Mises equivalent stress, \(\Sigma^\prime_{ij}\) the deviatoric stress tensor, \(\dot{\varepsilon}_{ij}\) the strain rate tensor and \(n\) is the stress sensitivity (power-law index) = \(1/(1 - n_c)\).

The bitumen tested by Cheung and Cebon (1997) was used to manufacture the idealised asphalt mixes for the experiments described below.

2.1.2. Idealised mixes
Two types of mixes (in the form of cylindrical specimens) consisting of bitumen mixed with different kinds of particulate inclusions were tested in this study:

1. 64% by volume sand particles between 300 and 600 µm, 4% air voids and
2. 75% by volume sand consisting of a mixture of equal quantities of sand particles between 150 and 300 µm, and 1.18 and 2.36 mm, 4% air voids.

For details of the specimen preparation procedure see Deshpande and Cebon (1998).

2.2. Experimental setup
A schematic diagram of the experimental setup is shown in Fig. 2. A steady axial load was applied to the specimen by a standard hydraulic testing machine through a submersible load cell as shown in Fig. 2. The submersible load cell measured

\[^2\] Pen refers to the “penetration grade” (Anon, 1989a) of the bitumen.
vertical load applied to the specimen and was insensitive to the fluid pressure within the triaxial cell. The axial strain of the specimen was determined from the load line displacement (i.e., the movement of the hydraulic actuator, which was measured using an LVDT as shown in Fig. 2). The radial strain was measured using an Hall effect radial strain transducer. Details of this transducer may be found in Clayton et al. (1989). The output of the radial strain transducer and the outputs of the submersible load cell and LVDT were recorded by a computerised data logging system.

The asphalt specimen was placed between two platens whose surfaces were lubricated with a mixture of glycerine and natural soap to reduce friction (and thus minimise bulging). A few tests were conducted with the specimen sealed from the pressurising fluid (water) by means of an impermeable rubber. However, this sealing was found to be unnecessary because of the low void fraction of the specimens. So in most tests the specimens were pressurised directly by the water. After the triaxial cell was pressurised to the required hydrostatic pressure, the axial load was applied rapidly and maintained at a constant value.

The force measured by the submersible load cell \( Q \) and the water pressure \( P \) are related to the principal stresses (see Fig. 3) by

\[
\Sigma_{33} = Q/A + P, \quad \Sigma_{22} = P, \quad \Sigma_{11} = P, \tag{5}
\]

where \( A \) is the cross-sectional area of the specimen. Thus, the mean stress \( \Sigma_m \) and a measure of deviatoric stress \( \Sigma \) are given by

\[
\Sigma_m = \Sigma_{kk}/3 = P + \frac{Q}{3A}, \tag{6}
\]

\[
\Sigma = \Sigma_{33} - \Sigma_{11} = \frac{Q}{A}. \tag{7}
\]
The triaxial tests were performed over a range of hydrostatic and deviatoric stress states. The deviatoric stress \( \Sigma \) was varied over approximately 3 orders of magnitude for each constant stress ratio \( \eta = \Sigma_m / \Sigma \). The stress ratio \( \eta \) was varied from \( \eta = 1/3 \) (uniaxial) to \( \eta = 1 \) (\( P = 2Q/3A \)) for the 64% mix and from \( \eta = 1/3 \) to \( \eta = 0.8 \) for the 75% mix. The reasons for choosing these stress ratios are discussed later in the paper.

2.3. Experimental results

Experimental observations from triaxial compression tests on the 64% and the 75% mixes are described in this section.

The variation of axial creep strain with time is shown in Fig. 4, for a typical triaxial creep test on the 64% mix. The creep curve can be divided into three regions: primary creep \((t < 15 \text{ s})\) where the strain rate decreases with time, secondary creep \((15 \text{ s} < t < 35 \text{ s})\) where the strain rate remains constant and tertiary creep \((t > 35 \text{ s})\) where the strain rate increases. The steady-state creep response of the material is defined as the secondary creep strain rate. The variation of the radial strain with time was observed to be qualitatively similar to the axial strain response.

Tests were performed over a wide range of hydrostatic and deviatoric stresses. The hydrostatic stress ranged from 24 to 1700 kPa, while the deviatoric stress ranged from 60 to 1700 kPa. The steady-state axial creep behaviour observed from triaxial tests on the 64% mixes is summarised in Fig. 5, where the steady-state axial strain rate is plotted against the deviatoric stress, \( \Sigma \). Also shown in Fig. 5 is the curve representing the uniaxial steady-state behaviour of bitumen at 20°C, obtained by Cheung and Cebon (1997). Since bitumen is a von Mises material (Eq. (4)), this curve will be the same for all axisymmetric compressive stress states. It can be seen that at each constant stress ratio \( \eta \), the curves for the asphalt mix have the same shape as the curve for pure bitumen at the same temperature. Thus, the steady-state triaxial behaviour can be represented by a modified version of Eq. (1):

\[
\frac{\sigma}{\sigma_0} = \frac{S\varepsilon}{\varepsilon_0} \left( \frac{1}{1 + (S\varepsilon/\varepsilon_0)^{n_c}} \right), \tag{8}
\]

where the “stiffening factor” \( S \) is a function of the stress ratio \( \eta \). From Fig. 5 it can be seen that \( S \) increases from approximately 1000 for \( \eta = 1/3 \) (uniaxial) to \( 2 \times 10^4 \) for \( \eta = 1 \). Fig. 6 shows the variation of volumetric strain

\[
H = E_{33} + 2E_{11}, \tag{9}
\]

with distortional strain

\[ S \] is the ratio of the steady-state deformation rate of bitumen to the steady-state deformation rate of the mix, subjected to the same deviatoric stress.
for all of the triaxial tests conducted on the 64%. The relationship between $H$ and $E$ can be represented simply by

$$H = s|E|,$$

(11)

where $s$ is the dilation gradient or slope of the line in Fig. 6. The slope $s$ varies between 0.75 and 0.85 for the 64% mix. This variation is likely to be due to the uncertainty in the diameter of the specimen, nonuniform deformation of the specimen (bulging) and the accuracy of the transducer. It is apparent that $s$ is independent of the stress ratio $\eta$, the deviatoric stress $\Sigma$ and the axial strain rate. It should be noted that the normal sign convention is that dilation is positive, hence the absolute value of the distortional strain is taken in Eq. (11).

Similar tests were conducted on the mix with 75% sand and the observed steady-state creep behaviour under various triaxial stress states is summarised in Fig. 7. The creep behaviour can also be represented by Eq. (8) with the stiffening factor $S$ being a function of the stress ratio $\eta$. Here too the stiffening factor increases with an increase in the stress ratio.

As for the 64% mix, a linear relationship was observed between the distortional and volumetric strains (Eq. (11)) with $s$ varying between 0.6 and 0.7 for the 75% mix. This relationship is again found to be independent of the stress ratio $\eta$, the deviatoric stress $\Sigma$ and the axial strain rate.

2.4. Discussion

While the 64% mix was tested to a maximum stress ratio $\eta = 1.0$, the 75% mix could only be tested to a maximum stress ratio of $\eta = 0.8$. Under high hydrostatic pressure (i.e., $\eta = 1.0$ for the 64% mix and $\eta = 0.8$ for the 75% mix), the specimens consistently failed along a weak plane. Various specimen making techniques were tried (e.g., the specimens were compacted in 5 layers, the compaction pressure was increased from 5 to 20 MPa) in an attempt to make more homogeneous specimens. However, it was found that above those critical stress ratios the specimens always tended to undergo localised deformation. Hence, tests at higher stress ratios did not give meaningful results.

The triaxial tests were only performed at room temperature (20°C). It is expected that the temperature dependence of the mix under triaxial stresses will be the same as that of bitumen (Deshpande and Cebon, 1998). The temperature sensitivity of bitumen is a result of the activation energy of the bitumen molecules or the free volume in the bitumen molecular structure (Eqs. (2) and (3)) and hence is not expected to be dependent on the stress state.
3. Constitutive relationship for idealised bituminous mixes

A bituminous mix, which consists of a high volume fraction of rigid aggregate in a bitumen matrix is in some ways similar to a soil which consists of soil solids and water. Moreover, the observed dilation of the idealised bituminous mixes is very similar to that seen in tests on soil samples. Hence, a model which is analogous to various soil mechanics models (e.g., the shear box model of Taylor (1948) and the Cam–Clay model, Roscoe et al., 1958) is developed in this section.

First, the analogy between the shear-box test on dry sands and the mechanisms controlling the deformation of the idealised bituminous mixes is discussed. The viscous dissipation rate is then approximately estimated using a modified version of the "composite sphere model" (Hashin, 1962). Finally, an overall constitutive relationship for the bituminous mix is derived using an equation analogous to that used to calculate the strength of a soil in the shear box test.

3.1. Shear box analogy

In the analysis of a shear box test on dry sand (see Fig. 8), Taylor (1948) proposed that two factors made contributions to the strength of the soil. One factor is the frictional resistance between particles as they slip during shear distortion of the soil. A second factor he called "interlocking"; this factor requires work to be done to cause volume increase during shear distortion. As an example, consider a shear box test on close-packed spheres (Fig. 8(a)). As the relative displacement of the two halves of the shear box is increased by $dx$, there is an increase in the sample thickness and a separation of the two halves of the box by an amount $dy$ because the spheres need to ride up on each other. This represents the internal kinematic constraints imposed by the rigid spheres. The quantity $dy/dx$ indicates the dilation of the spheres in the shear zone. It is a measure of the interlocking. In soil specimens $dy/dx$ tends to zero at large shear strains, and the two halves of the sand body in the shear box slip relative to each other without further dilation. This is shown in Fig. 8(b). A very thin layer of loosened sand on the slip plane is then said to have reached a critical state and there is no more interlocking. A work equation for the shear box test can be written as (Taylor, 1948)

$$\tau \, dx = \mu \sigma \, dx + \sigma \, dy$$

work in = friction dissipation + dilation work

or

Fig. 8. Shear box analogy.
\[ \tau/\sigma = \mu + dy/dx \]

strength = friction + interlocking. \hspace{1cm} (13)

The idealised bituminous mix can also be schematically represented by Fig. 8(a), with bitumen filling the gaps between the rigid particles. Assuming no frictional contact between particles (i.e., full “lubrication” of the particles by the binder), an analogous work equation for the idealised bituminous mix is

work in = viscous dissipation + dilation work. \hspace{1cm} (14)

The relation between the shear and volume strains is governed by the kinematic constraints imposed by the rigid sand particles. In the following section a brief outline of the current understanding of the quasi-static mechanics of granular materials is given.

3.2. Quasi-static mechanics of granular assemblages

The quantity dy/dx, which indicates the dilation in the two dimensional analogy, is governed primarily by the shape, size, and gradation of the sand particles (i.e., the geometry of the aggregate). Numerous researchers have studied the quasi-static mechanics of granular assemblages. A brief outline of the relevant literature is given below. A more detailed review is given in Deshpande (1997).

Reynolds (1885) first revealed, conceptually and experimentally, the phenomenon of volume-coupled shape change. An outline of a continuum theory of Reynolds dilatancy is given by Goddard and Bashir (1990). They asserted that Reynolds dilatancy must be interpreted as an internal kinematic constraint reflecting the “steric” or geometrical effects which operate in the quasi-static motion of nearly rigid granules. In the absence of any such internal constraint, the volume or density of a compressible material is independent of its shape; i.e., volumetric and shear strains represent independent kinematic quantities.

As a simplest generalisation of Reynolds infinitesimal-strain theory to arbitrary states of deformation, Goddard and Bashir (1990) expressed the nonholonomic \(^4\) internal constraints in a form applicable to any type of deformation

\[ \varepsilon_v = \alpha |E|, \hspace{1cm} (15) \]

where \(|E|\) is \(\sqrt{\text{tr}(E^2)}\); \(E\) the strain tensor; \(\alpha\) the coefficient of dilatation (it is generally dependent on the entire history of deformation); “\(\text{tr}\)” the trace of the tensor and \(\varepsilon_v\) the volumetric strain (the 2-D equivalent of \(H\)).

As an example of constrained deformation, consider the case of dilatant simple shear (in 2-D), involving simultaneous isotropic expansion. The components of \(E\) are given by

\[ E = \frac{1}{2} \begin{pmatrix} \varepsilon_v & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \varepsilon_v \end{pmatrix}, \hspace{1cm} (16) \]

where \(\gamma\) is the 2-D analogue of the distortional or shear strain \(E\). Hence, Eq. (15) gives

\[ s = \frac{\varepsilon_v}{\gamma} = \frac{\alpha}{(2 - \alpha^2)^{1/2}}. \hspace{1cm} (17) \]

Numerous researchers (e.g., Bashir and Goddard, 1991; Zhuang et al., 1995) have used numerical simulations to model the deformation of 2-D and 3-D random arrays of rigid spheres and found a linear dependence of the volumetric strain on the shear strain, with values of \(s\) ranging from 0.7 to 1.8.

An important consequence of the existence of an internal constraint is that it leads to rheologically indeterminate stresses, as discussed in Truesdell and Noll (1965). Goddard and Bashir (1990) showed that subject to an internal constraint, the stress tensor \(\Sigma\) is rheologically determine only up to an additive stress \(\Sigma^0\), which does no work in any deformation compatible with the constraint (i.e., adding a stress \(\Sigma^0\) to the applied stress will not affect the deformation).

For example, the hydrostatic or mean stress is indeterminate and does no work when the material is subjected to an incompressibility constraint. Another illustration is the case of dilatant simple shear (Eq. (16)):

\(^4\) Constraints on kinematics which are not generally derivable from constraints on static configurations.
\[ E = \frac{\gamma}{2} \begin{pmatrix} s & 1 \\ 1 & s \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_{11} & \tau \\ \tau & \sigma_{22} \end{pmatrix}, \]  

where \( \tau \) is the shear stress and the mean stress \( p = (\sigma_{11} + \sigma_{22})/2 \). Goddard and Bashir (1990) showed that the indeterminate stress state is given by

\[ \Sigma^0 = \lambda \begin{pmatrix} -1 & s \\ s & -1 \end{pmatrix}, \]  

where \( \lambda \) is a proportionality constant. Thus, a family of stresses given by

\[ \tau = \tau_o + \lambda s, \]

\[ p = p_o - \lambda, \]

are rheologically indistinguishable. The terms \( \lambda s \) and \( -\lambda \) are the additive shear and mean stresses, respectively.

While the strains are independent for unconstrained materials, they are connected by the relation \( \epsilon_v = s \gamma \) for the dilatant material. Thus, the work done, given by \( W = \Sigma \cdot E \), is

\[ W = \tau \gamma + p \epsilon_v = (\tau + sp) \gamma. \]  

The term \( \tau \gamma \) represents the shear ("shape") work and \( p \epsilon_v \) the volume work. The part of the stress which does the work of deformation is given by \( \tau + sp \). Hence, any stress state of the form given by Eq. (19) (i.e., \( \tau = \lambda s \) and \( p = -\lambda \)) is work free.

From the above arguments it follows that all energy input (Eq. (22)) into a granular material composed of rigid frictional particles for which kinetic energy and elastic strain energy are negligible must entirely be accounted for by dissipation associated with inter-particle sliding friction. Therefore, it follows that for the limiting case of ideal frictionless particles, \( W \) must vanish and the stress state takes the form given by Eq. (19) with all the stresses representing work free reactions.

### 3.3. Viscous dissipation rate

Deshpande and Cebon (1998) conducted uniaxial compression tests on bituminous mixes with volume fractions of sand ranging from 40% to 85%. They found that the “stiffening factor” \( S \) increased with volume fraction as shown in Fig. 9. Further, they found that the low volume fraction mixes (40% and 52% mixes) deformed at constant volume (no dilation) while the high volume fraction mixes (64%, 75% and 85%) dilated under a uniaxial compressive stress.

Deshpande and Cebon (1999) modelled these mixes as two phase composites consisting of rigid inclusions in a nonlinear viscous matrix. They showed that the Hashin composite sphere model upper bound, modified to the nonlinear viscous case, accurately predicted the stiffening factors over the entire range of volume fractions tested (Fig. 9). However, the composite sphere model does not account for any mechanism that causes the high volume fraction mixes to dilate under a compressive stress state and hence is not a complete model for the overall constitutive behaviour of the dense mixes. Such a complete constitutive relationship is derived in Section 3.4 using an equation analogous to Eq. (14). It is shown that for a mix subjected to a uniaxial compressive stress the major contribution to the stiffening factor is from the viscous dissipation rate and the other term (i.e., volume work) is small. Since the modified composite sphere model upper bound accurately predicts the stiffening factors under uniaxial compression for the fully dense mixes it is expected to provide a reasonable estimate of the viscous dissipation rate in these mixes for other stress states as well.

The composite sphere model proposed by Hashin utilises a unit cell consisting of a spherical inclusion surrounded by a certain quantity of the
matrix associated with that inclusion. In granular materials like the fully dense bituminous mixes, it is expected that each aggregate particle will be coated with a thin layer of bitumen and there will be particle to particle contact (as shown in Fig. 8(a)) through these films. Thus, Hashin’s composite sphere model is not strictly applicable to these mixes, nevertheless it does estimate the viscous dissipation in these mixes with sufficient accuracy.

Consider a composite with a volume fraction \( c \) of rigid inclusions and \( v \) of voids, in an incompressible linear elastic matrix. An upper bound on the shear modulus \( \mu \) of this composite using Hashin’s composite sphere model is (Hashin, 1962)

\[
\frac{\mu}{\mu_m} = \left[ 1 + \frac{c}{2(1 - c)} \left( 1 - \frac{\rho_1}{\rho_2} \right) \right] \frac{v}{\frac{2}{3} v + 2} + \frac{\rho_1 - \rho_2}{\rho_2},
\]

where \( \mu_m \) is the shear modulus of the matrix material. However, the composite sphere upper bound on the bulk modulus of a three phase composite consisting of voids and rigid inclusions in an elastic matrix is trivial (i.e., \( +\infty \)).

Christensen and Lo (1979) attempted to get exact solutions (not bounds) for the composite sphere morphology using a self-consistent scheme (generalised self-consistent model, “GSC”). The bulk modulus predictions of the GSC model for the two phase voided composite coincide with Hashin’s solution. Hence, the GSC model is used here to estimate the bulk modulus of the three phase composite. Huang et al. (1994) extended the GSC model to handle the \( n \)-phase case. The bulk modulus of a three phase composite consisting of a dispersion of voids and rigid inclusions in an incompressible linear elastic matrix is

\[
k = \frac{4}{3} \frac{1 - \nu}{1 - c} \mu_m.
\]

With these expressions for the bulk and shear modulus the energy dissipation rate in a linear elastic or viscous composite can be estimated. However, the bituminous mixes tested are non-linear viscous for most practical stress levels. Hence, the dissipation rate needs to be calculated using the nonlinear equivalent of the composite sphere model.

Suquet (1993) proposed a method to convert upper bounds for voided or rigidly reinforced linear composites to upper bounds for nonlinear viscous composites with the same microstructure. Deshpande (1997) made a minor modification to this theory to enable it to be used for three phase voided and rigidly reinforced composites. This theory is used here to transform the linear composite sphere solutions.

Using Suquet’s transformation, an upper bound on the dissipation rate \( D \) in the three phase composite under consideration is

\[
D \leq \sigma_0 \epsilon_0 \left( \frac{\dot{E}_{eq}}{\dot{E}_c} \right)^{(n+1)/n}
\]

with

\[
\dot{E}_{eq} = \left( \dot{W}_o (E)(1 - c - v)^{(n-1)/(n+1)} \right)^{1/2},
\]

where

\[
\dot{W}_o = \frac{1}{2} \left( k \dot{H}^2 + 3 \mu \dot{E}_c^2 \right).
\]

The parameters \( k \) and \( \mu \) are calculated using Eqs. (24) and (23), respectively and \( \mu_m = 2/3 \) (see Suquet (1993) for the reason behind this choice of \( \mu_m \)). \( \dot{E}_c = \left( \dot{E}_{ij} \dot{E}_{ij} \right)^{1/2} \) is the von Mises equivalent strain rate and \( H = \dot{E}_{kk} \) is the hydrostatic strain rate. \( \dot{E}_{ij} \) is the deviatoric strain rate tensor.

Deshpande and Cebon (1999) checked this transformation numerically for a rigidly reinforced nonlinear viscous composite modelled by the Hashin composite sphere model. A comparison between the stiffening factors observed in uniaxial compression experiments, the numerical calculations and the transformed Hashin composite sphere upper bound is shown in Fig. 9. Suquet’s method overestimates the stiffening factor by about an average of 45% for the bitumen composite over the entire range of six orders of magnitude of \( S \). Thus, the analytical expression for the dissipation rate can be corrected using a scaling factor obtained from the numerical calculations. Note that the void fraction \( v \) is ignored while comparing the experimental results with the models. The void fractions in the three phase
composite under consideration here are very small compared to the volume fraction of the aggregate and have a negligible effect on the dissipation rate or stiffening factors.

### 3.4. Overall constitutive relationship

An overall constitutive law for the steady-state viscous behaviour of idealised bituminous mixes is developed in this section based on a formulation analogous to the shear box example. All the energy dissipation is accounted for by viscous dissipation in the bitumen. This can be written as

\[
\Sigma_m \dot{H} + \Sigma_c \dot{E}_c = D,
\]

where \( \Sigma_m \) represents the volume work, while \( \Sigma_c \dot{E}_c \) the shear or “shape” work. From the discussion about the dissipation rate \( D \) in the previous section, Eq. (28) can be rewritten using Eqs. (25)–(27) as

\[
\Sigma_m \dot{H} + \Sigma_c \dot{E}_c \leq \frac{\sigma_o}{\varepsilon_o^{1/n}} \left( \frac{1}{2} k \dot{H}^2 + \frac{3}{2} \mu \dot{E}_c^2 \right)^{(n+1)/2n}
\times \left[ 1 - (c + v) \right]^{(n-1)/2n}.
\]

It was seen from the uniaxial and triaxial compression tests on the idealised mixes that the volumetric strain is related to the distortional strain (to include non-axisymmetric cases) as

\[
H = s \dot{E}_c.
\]

Since this relation is independent of stress, strain rate, etc., the rate form of this equation is

\[
\dot{H} = s \dot{E}_c.
\]

Substituting Eq. (31) into Eq. (29) and rearranging gives an overall constitutive relationship for the idealised mixes

\[
\dot{E}_c \geq \frac{\dot{\varepsilon}_o (\cos \omega + 1)^n}{\left( \frac{1}{2} k s^2 + \frac{3}{2} \mu \right)^{(n+1)/2} \left[ 1 - (c + v) \right]^{(n-1)/2}} \left( \frac{\Sigma_c}{\sigma_o} \right)^n,
\]

where \( \omega = \Sigma_m / \Sigma_c \). The volumetric strain rate \( \dot{H} \) is given by Eq. (31).

Thus, Eq. (32) along with Eq. (31) characterises the steady-state deformation behaviour of the material using the properties of bitumen (i.e., \( \dot{\varepsilon}_o, \sigma_o \) and \( n \)) and the aggregate packing properties \( c, v, s \). While the volume fractions of the aggregate \( c \) and voids \( v \) are known and used to calculate \( k \) and \( \mu \), the “dilation gradient” \( s \) must be estimated. Some numerical studies, such as those discussed in Section 3.2, have attempted to calculate \( s \). However, present theoretical knowledge of the deformation of granular materials is insufficient to estimate \( s \) for the practical cases of a gradation of non-spherical aggregate particles. Hence, \( s \) is taken as an experimental input in the present model.

As the mix deforms, it dilates and thus the porosity increases. The evolution law for the porosity takes a simple form for an incompressible matrix (Gurson, 1977)

\[
\dot{s} = (1 - v) \dot{H}.
\]

The volume fraction of the aggregate evolves due to the overall volume change according to

\[
\dot{c} = c \dot{H}.
\]

The evolution of the mix properties with deformation is therefore given by Eqs. (32)–(34). Note that only the first order effects of the porosity evolution are taken into account here (i.e., the change in volume fraction of the voids is considered, but any anisotropy due to change in aspect ratio of voids is ignored). Also, the voids are assumed to be sufficiently well dispersed that the local dilation does not generate additional stresses.

#### 3.4.1. Axisymmetric loading

The above constitutive relationship can be applied to axisymmetric cases like the uniaxial and triaxial experiments described earlier. In this case, the deviatoric stress \( \Sigma \), and the distortional strain rate \( \dot{E} \) used to describe the observed deformation behaviour of the idealised mixes are related to \( \Sigma_c \) and \( \dot{E}_c \) by

\[
|\Sigma| = \Sigma_c \quad \text{and} \quad |\dot{E}| = \dot{E}_c.
\]

The shear or “shape” work \( \Sigma \dot{E} \) is always positive as it represents the dissipative work. Hence, \( \text{sign}(\Sigma) = \text{sign}(\dot{E}) \). Thus, the general constitutive relationship in the axisymmetric case can be rewritten as
\[\dot{E} = \frac{\dot{\varepsilon}_0 [\eta s \text{sign}(\Sigma) + 1]^n}{\left(\frac{1}{2} k s^2 + \frac{3}{2} \mu\right)^{(n+1)/2} [1 - (c + v)]^{(n-1)/2} \left(\frac{1}{\sigma_0}\right)^n \times \text{sign}(\Sigma)},\]

where the \(\geq\) sign has been replaced by the \(=\) sign for convenience. However, it should always be remembered that this is an upper bound estimate.

The axial strain rate \(\dot{E}_{33}\) is given by

\[\dot{E}_{33} = \dot{\varepsilon}_0 \left(1 + \frac{s}{S} \text{sign}(\Sigma)\right)\]

Substituting for \(\dot{E}\) using Eq. (36), the axial strain rate can be written as

\[\dot{E}_{33} = \frac{\dot{\varepsilon}_0}{S} \left(\frac{1}{\sigma_0}\right)^n \text{sign}(\Sigma),\]

where the stiffening factor \(S\) is given by

\[S = \left(\frac{1}{2} k s^2 + \frac{3}{2} \mu\right)^{(n+1)/2} [1 - (c + v)]^{(n-1)/2} \left[\eta s \text{sign}(\Sigma) + 1\right]^n [1 + \frac{1}{2} s \text{sign}(\Sigma)].\]

Thus, the overall constitutive relationship has the same form as the uniaxial and triaxial experimental results with the stiffening factor being a function of the volume fraction of the aggregate \(c\) and voids \(v\), and the stress ratio \(\eta\).

The modified Hashin composite sphere model was used to estimate the viscous dissipation rate in the above formulation. The driving force behind this choice was the fact that the composite sphere model predicted the experimentally observed stiffening factors correctly for the uniaxial compression case; a stress state in which the volume or dilation work term was expected to be small. The accuracy of this assumption can be checked by switching off volume work in Eq. (36) (i.e., set \(s = 0\)). Thus, for the uniaxial compression case \((\eta = 1/3)\), the error is expected to be

\[\frac{S_{s=0.6} - S_{s=0}}{S_{s=0.6}} \approx 52\%.\]

An error of approximately 50% is within the experimental scatter associated with random composites like asphalt where manufacturing conditions, compositions and micro-structures greatly differ (see Figs. 12 and 13). It can be compared to the five orders of magnitude variation of \(S\) measured in the experiments (see Fig. 9).

### 3.4.2. An illustration of the use of the constitutive law

A flow diagram summarising the procedure for calculating the mix properties as it deforms is shown in Fig. 10. As an example consider a mix with 64% by volume aggregate and 4% by volume voids, subjected to triaxial compressive stresses. The dilation gradient \(s\) is measured to be approximately 0.8 (Fig. 6).

The moduli \(k\) and \(\mu\) are calculated using Eqs. (24) and (23) and the stiffening factor \(S\) is then estimated using Eq. (39) for a particular stress ratio \(\eta\) with sign \(\Sigma\) taken as negative for triaxial compression \(^5\) by convention. The axial deformation rate can then be estimated from Eq. (38) (or equivalently the distortional strain rate from Eq. (36)) and the volumetric strain rate calculated using \(\dot{H} = s|\dot{E}|\).

The distortional deformation \(E\) is then evaluated using simple Euler integration. The resulting volumetric strain due to the kinematic constraint is

---

\(^5\) This model is expected to be valid only under compressive stresses as explained in Section 5.
estimated from $H = s[E]$. The evolution Eqs. (33) and (34) are also integrated to calculate the new volume fractions of voids and aggregate.

The new values of $c$ and $v$ are then used to re-calculate values of $k$ and $\mu$ and the entire procedure is repeated until the required strain value (see Fig. 10). The variation of the stiffening factor $S$ with axial strain is shown in Fig. 11 for various values of the stress ratio $\eta$. It can be seen that the stiffening factor decreases with strain. This is because the mix dilates and increases the void fraction as it deforms.

3.4.3. Stress indeterminacy

As discussed in Section 3.2, the existence of internal constraints leads to rheologically indeterminate stresses. The kinematic constraint on the deformation of the idealised mixes, $H = sE_e$, is a generalisation of the two dimensional constraint $e_v = s\gamma$, discussed by Goddard and Bashir (1990). Hence, as argued by Goddard and Bashir (1990), the stresses in this case will be determinate only to an additive stress

$$\Sigma_e = s\lambda \quad \text{and} \quad \Sigma_m = -\lambda,$$

where $\lambda$ is a proportionality constant. This implies that for the same the deformation rates (or the strain rate tensor) there exists a family of stresses given by

$$\Sigma_e = \Sigma_{eo} + s\lambda$$

$$\Sigma_m = \Sigma_{mo} - \lambda,$$

which satisfy the constitutive relationship. By setting $\lambda = 0$ in the above equations, it is seen that $\Sigma_{eo}$ and $\Sigma_{mo}$ are constants chosen to satisfy the constitutive relationship for the given deformation rate. An illustration of this stress indeterminacy is given in Section 4.

4. Comparison with experimental results

In this section the predictions of the constitutive model are compared with the uniaxial and triaxial compression experimental observations.

This model is not applicable to the entire deformation history, as it only considers the steady-state viscous behaviour of bitumen (not the initial elastic response). Hence, this model is expected to be appropriate only when the viscous behaviour of the bitumen is dominant. It is observed in the uniaxial as well as triaxial experiments that the steady-state viscous behaviour of the 64% mix is reached at around 4% axial strain. Therefore, the stiffening factor predicted by the model at 4% axial strain is compared with the observed stiffening factors in the 64% mixes. The variation of the stiffening factor $S$ with the stress ratio $\eta$ when the 64% mix is subjected to triaxial compressive stresses is shown in Fig. 12 for three values of $s$. As discussed earlier, there is some uncertainty in the value of $s$ due the non-uniform deformation of the specimen (bulging), the uncertainty in the diameter
of the specimen and the accuracy of the radial strain transducer. As a consequence $s$ was observed to be between 0.76 and 0.85. From the experimental results plotted (with error bars) in Fig. 12, it can be seen that $s = 0.78$ agrees well with the entire set of experimental data. This value is within the range of the experimentally measured values of $s$ which indicates that the model fits the measured data well.

The stiffening factor curves shown in Fig. 12 rise dramatically above a certain value of $\eta$, depending on the value of $s$. This means that the "stiffness" of the mix increases substantially and deformation rates of the idealised mixes are expected to be very small at those high values of $\eta$. The reasons and practical implications of this are discussed later in the paper.

A similar analysis can also be performed for the 75% mix. The steady-state viscous behaviour of the 75% mixes was observed around 3% axial strain. Hence, the stiffening factors predicted by the constitutive model at 3% axial strain should correspond to the observed stiffening factors. Fig. 7 shows the variation of the stiffening factors (at 3% axial strain) with the stress ratio $\eta$ when the 75% mix is subjected to triaxial compressive stresses. From the figure it can be seen that $s = 0.6$ best agrees with the experimental results, and since this is within the range of the experimentally measured values of $s$ ($s = 0.6 - 0.7$), it again confirms the accuracy of the model. Using the numerical correction to Suquet’s equation discussed in Section 3.3 (Fig. 9), a modified constitutive relationship is derived. It differs from the original law by a constant factor of 15% (the stiffening factor $S$ is overestimated using Suquet’s transformation by approximately 45% while the dissipation rate $D$ is overestimated by $S^{1/\alpha}$, hence approximately 15%). Suquet’s prescription gives an analytical upper bound expression for the viscous dissipation rate. The 15% correction added to the modified model takes it nearer the optimum upper bound. The dashed lines in Figs. 12 and 13 show the corrected stiffening factors for $s = 0.78$ and 0.6, respectively. It is recognised that this correction is not strictly valid as $v$ is not equal to zero in the bituminous mixes tested and the correction ignores the viscous dissipation due to the dilation (the bulk modulus term in dissipation equations). However, it gives an indication of amount by which the viscous dissipation rate and hence the stiffening factors are overestimated by using Suquet’s analytical expressions.

4.1. Stress indeterminacy

As explained in the previous section, the existence of internal kinematic constraints leads to a stress indeterminacy, and hence a family of stresses given by Eqs. (42) and (43) satisfy the constitutive law for the same deformation rate. An illustration of this is given in Fig. 5, where the steady-state deformation behaviour of the 64% mix is shown for various compressive triaxial stress states. Four stress states 0, 1, 2 and 3, as illustrated in the figure, result in the same axial deformation rate. It is however clear from the kinematic constraint that the distortional and volumetric strain rates will also be the same in each case. Thus, given only the deformation rate tensor, the stress state in the specimen cannot be specified uniquely. For example, in Fig. 5, four stress states result in an axial deformation rate of $10^{-4}$ per second. The values of the stresses in the stress states “1”, “2” and “3” can be derived from the stress state 0 using Eqs. (42) and (43) with $\lambda \approx 1.6 \times 10^5$ Pa, $\lambda \approx 4.1 \times 10^5$ Pa and $\lambda \approx 7.9 \times 10^5$ Pa, respectively (the dilation gradient $s$ was taken to be 0.78, consistent with the earlier discussion). Thus, given the stress state $\Sigma_{mo}$ and $\Sigma_{o}$, an entire family of
stresses giving the same deformation rate but different values of \( \eta \) can be chosen by using different values of \( \lambda \).

5. Discussion

An interpretation of the various terms in the constitutive relationship, Eq. (32), is given in the following paragraphs.

Consider the shear box in Fig. 8. If the normal stress \( \sigma \) is increased while keeping the shear stress \( \tau \) constant, a point will be reached where no further deformation occurs. This is because for a displacement in the \( x \) direction to occur, the spheres need to ride up on each other, which in turn results in a displacement in the \( y \) direction. The compressive stress \( \sigma \) does negative work which must be provided by the positive work done by the shear stress (hence the dependence of the behaviour on the mean stress). Beyond a certain critical stress ratio \( \sigma/\tau \) no further deformation will take place: “lockup” due to the interlocking of the aggregate occurs. This lockup will only occur when the normal stress \( \sigma \) is compressive. This phenomenon is reflected in the term \( [\cos 1]^n \) in the constitutive relationship, Eq. (32). When the mean stress \( \Sigma_m \) is compressive, \( \cos \alpha \) is negative and thus as the absolute value of the stress ratio \( \cos \alpha \) increases, \( [\cos 1]^n \) decreases. As \( |\cos \alpha| \) increases the deformation rate decreases or the stiffening factor increases as seen from Figs. 12 and 13. When the stress ratio reaches a critical value given by \( \cos \alpha = -1/s \), the term \( [\cos 1]^n = 0 \) and lockup occurs, i.e., no deformation is possible. The lockup of the mixes is controlled by the dilation gradient \( s \) of the aggregate. The 64% and the 75% mixes are expected to lockup at \( |\cos \alpha| \) or \( \eta = 1.28 \) and 1.67 for \( s = 0.78 \) and \( s = 0.6 \), respectively. This is indicated by the asymptotes sketched in Figs. 12 and 13.

Figs. 12 and 13 show that increasing the dilation gradient \( s \) results in a higher stiffening factor for a mix with the same volume fraction of aggregate subjected to the same stress ratio. Thus, increasing the dilation gradient not only results in earlier lockup but also an increase in the strength of the bituminous mixes. Hence, \( s \) is a critical factor which should be taken into consideration while designing aggregates for bituminous mixes. This is not done in any known asphalt mix design procedure.

The relationship between \( s \) and aggregate properties like particle shape, size gradation and volume fraction is not understood. Since \( s \) has such a dominant effect on the deformation properties of the mix, further work into understanding the factors affecting \( s \) needs to be undertaken to improve aggregate design.

The denominator in the constitutive law, Eq. (32)

\[
\left( \frac{1}{2} k \alpha^2 + \frac{3}{2} \mu \right)^{(\eta+1)/\eta} \left[ 1 - (c + e) \right]^{(\eta-1)/2},
\]

is the viscous dissipation term. It is calculated from the nonlinear viscous composite sphere model and thus is only a function of the stress exponent of the bitumen \( n \) and the volume fractions of the aggregate and the voids. As expected, it increases with the volume fraction of the aggregate. Hence, the strength or stiffening factor of the mix increases with the volume fraction of aggregate.

The constitutive law, Eq. (32), is in a form that could be applied to all stress states. However, the model does not take into account mechanisms like fracture of the bitumen, and coalescence and growth of voids in the bituminous mixes. These mechanisms are expected to be important when the mix is subjected to tensile stresses (Genin and Cebon, 1998b). Hence, this constitutive law should only be applied to bituminous mixes subjected to compressive stresses (as in all the experimentation on mixes described in this work).

The model described here only takes into consideration the steady-state viscous behaviour of bitumen and ignores the elastic and transient effects. Thus, this model only becomes applicable when the viscous behaviour of bitumen is dominant (i.e., in the steady-state or secondary creep regimes of the mix deformation). It is known that the initial elastic and transient behaviour of bituminous mixes is important in understanding the response of a pavement under the load pulse generated by moving wheel loads. It may be possible to modify the present model to account for...
this behaviour by developing appropriate constitutive relationships for the elastic and transient behaviour of bitumen. Therefore, the model in its present form only represents a first step towards more complete micro-mechanical modelling of bituminous mixes.

6. Conclusions

1. A model was developed to characterise the steady-state deformation behaviour of idealised bituminous mixes under compressive stress states. It is analogous to some soil mechanics models in that it assumes that the volumetric and distortional strains are constrained by the kinematics of the aggregate. It shows that the deformation behaviour of the bituminous mixes is dependent on the mean and deviatoric stresses.

2. The model predicts that the stiffening factor of mixes subjected to compressive stresses increases with the stress ratio $\eta$. Above a critical stress ratio (governed by the dilation gradient $s$), the mixes are expected to lockup (i.e., no deformation is possible).

3. The variation of the stiffening factor with the stress ratio $\eta$ was found to agree well with measurements from uniaxial and triaxial compression tests.

4. The existence of the internal constraints leads to a stress indeterminacy. This indeterminacy was checked against the experimental measurements and was found to agree with the model predictions.

Acknowledgements

The authors would like to express their gratitude to the Cambridge Commonwealth Trust and the UK Engineering and Physical Sciences Research Council for financial support for this work. The authors would also like to thank Prof. M.F. Ashby and Prof. A. N. Schofield of the Cambridge University, Engineering Department, for many helpful discussions.

References


Reynolds, O., 1885. On the dilatancy of media composed of rigid particles in contact. Philosophical Magazine and Journal of Science 20 (127), 469.


